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About the minimal time crisis problem and applications*

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Abstract We study the optimal control problem where the cost functional to be minimized represents the so-called time of crisis, i.e. the time spent by a trajectory solution of a control system outside a given set K. Such a problematic finds applications in population dynamics, such as in prey-predator models, which require to find a control strategy that may leave and enter the crisis domain K a number of time that increases with the time interval. One important feature of the time crisis function is that it can be expressed using the characteristic function of K that is discontinuous preventing the use of the standard Maximum Principle. We provide an approximation scheme of the problem based on the Moreau-Yosida approximation of the indicator function of K and prove the convergence of an optimal sequence for the approximated problem to an optimal solution of the original problem when the regularization parameter goes to zero. We illustrate this approach on a simple example and then on the Lotka-Volterra prey-predator model.

Keywords: Optimal control, Hybrid Maximum Principle, Viability, Prey-predator model.

1. THE MINIMAL TIME CRISIS PROBLEM

We consider a controlled dynamics in $\mathsf{X} \subset \mathbb{R}^n$

$$\dot{x}(t) = f(x(t), u(t)) \quad \text{a.e. } t \in [0, T], \tag{1}$$

a set of admissible controls

$$\mathcal{U} := \{u : [0, T] \to U \; ; \; u \text{ meas.}\},$$

where U is a compact convex set in $\mathbb{R}^m$. Let $K \subset \mathsf{X}$ be a closed set with non empty interior. The system is said to be in a *crisis* when $x(t)$ does not belong to the set K. We shall also consider the usual hypotheses, that we recall later, which guarantee the following assumption to be fulfilled.

Assumption 1. Given $t_0 \in (-\infty, T]$, $x_0 \in \mathsf{X}$ and $u(\cdot) \in \mathcal{U}$, we denote by $x_u(\cdot)$ the unique absolutely continuous solution of (1) such that $x(t_0) = x_0$ and defined over $[t_0, T]$.

We recall the following definitions from the Viability theory [1].

a) Viability kernel:

$$Viab(K) := \{x_0 \in K \; ; \; \exists u \in \mathcal{U}, \; x_u(t) \in K, \; \forall t \geq 0\}$$

b) Finite horizon viability kernel:

$$Viab_{[0,T]}(K) := \{x_0 \in K \; ; \; \exists u \in \mathcal{U}, \; x_u(t) \in K, \; \forall t \in [0, T]\}.$$

When the constraints set K is not viable (i.e. $Viab(K) \neq K$), and $x_0 \notin Viab_{[0,T]}(K)$ the trajectory spends *some time* outside K. One may then consider the *minimal time crisis* problem:

$$(\mathcal{P}) : \; J^T(u) := \int_{t_0}^{T} \mathbb{1}_{K^c}(x_u(t)) \, \mathrm{d}t \quad \to \inf_{u \in \mathcal{U}} J^T(u) \tag{2}$$

$$\text{where } \mathbb{1}_{K^c}(x) := \begin{cases} 0 & x \in K, \\ 1 & x \notin K. \end{cases}$$

Remark that trajectories may enter and leave K several times (possibly periodically). Let us recall some previous works related to this problem :

1. In [4, 5], linear parabolic equations (related to steel continuous casting model) are considered with the criterion

$$\sup\{t \; ; x_u(t) \in K\} - \inf\{t \; ; \; x_u(t) \notin K\}$$

A regularization method is proposed but considering only one crossing time from K to K^c.

2. In [8], the minimal time crisis problem is considered over an infinite horizon, and the value function is characterized as the smallest positive lower semi-continuous viscosity super-solution of

$$\begin{aligned} &H(x, \nabla V(x)) = 0, \; x \in \mathsf{X}, \\ &V(x) = 0, \quad x \in \partial Viab(K). \end{aligned}$$

Then, an approximation scheme of the epigraph of the value function with the discrete viability kernel algorithm (considering an augmented dynamics) is used. No necessary conditions are given for this problem.

Our objectives in the present work are

(1) to consider finite horizon,
(2) to do not fix any a priori numbers of entry and exit times, and to provide necessary optimal conditions,
(3) to setup a numerical scheme that could be used to approximate optimal trajectories on concrete problems.

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The paper is organized as follows. In the second section, we recall the existence of an optimal control for the time crisis problem and we apply the hybrid maximum principle provided that an optimal trajectory satisfies a transverse condition at every crossing time of K. The third section is devoted to the study of a regularization scheme of the time crisis problem and to the convergence of an optimal solution of the regularized problem to a solution of the time crisis problem. In particular, no transverse condition is required when studying the approximated optimal control problem. Finally, we consider in the last section two examples where the time crisis is of particular interest. In the first one, we show that a so-called *myopic* strategy is optimal : it consists in minimizing the time in K^c and in maximizing the time in K. In this example, the viability kernel is empty. The second example is based on the Lotka-Volterra model and involves a viability kernel. We first compute this set and provide an optimal synthesis for the minimum time problem to reach the viability kernel. Finally, we show that there exist initial conditions for which the time crisis is strictly less than the minimum time to reach the viability kernel.

2. EXISTENCE RESULT AND HYBRID MAXIMUM PRINCIPLE

Let us first state the results about the existence of optimal solutions, under the following hypotheses.

Hypothesis 2. f, U and K fulfill the following properties:

(1) The set U is a non-empty compact set of $\mathbb{R}^m$.
(2) The dynamics f is continuous w.r.t. (x, u), locally Lipschitz w.r.t. x and satisfies the linear growth condition: $\exists\, c > 0$ s.t. $\forall\, (x, u) \in \mathbb{R}^n \times U$, one has:
$$\|f(x, u)\| \leq c(1 + \|x\|).$$
(3) For any $x \in \mathbb{R}^n$, the set $F(x) := \{f(x, u)\ ;\ u \in U\}$ is a non-empty convex set.
(4) The set K is a compact set in $\mathbb{R}^n$ with non-empty interior.

Proposition 3. For any $t_0 \in (-\infty, T]$ and $x_0 \in \mathbb{R}^n$, there exists an optimal control of problem $(\mathcal{P})$.

Proof. As a sketch of proof, one can consider the *extended* set-valued map G from $\mathbb{R}^{n+1}$ into the subsets of $\mathbb{R}^{n+1}$:
$$G(z) := \left| \begin{array}{ll} f(x,U) \times \{0\} & \text{if } x \in \mathrm{Int}(K), \\ f(x,U) \times [0,1] & \text{if } x \in \partial K, \\ f(x,U) \times \{1\} & \text{if } x \in K^c, \end{array} \right.$$
where $z = (x, y)$, and use classical compactness arguments.

Define now the Hamiltonian associated to the optimal control problem:
$$H(x, p, p_0, u) = p \cdot f(x, u) - p_0 \mathbb{1}_{K^c}(x)$$
and notice that it is discontinuous w.r.t. to x preventing the use of the standard Maximum Principle. Instead, one may consider the Hybrid Maximum Principle (HMP). For this, we recall the definition of crossing times.

Definition 4. We say that a time $t_c \in [t_0, T]$ is a *regular crossing time* for a given trajectory $x(\cdot)$ if:

(1) The point $x(t_c)$ is s.t. $x(t_c) \in \partial K$, and there exists $\eta > 0$ such that for any $t \in [t_c - \eta, t_c)$, resp. $t \in (t_c, t_c + \eta]$, one has $x(t) \in K$, resp. $x(t) \in K^c$.
(2) The control u associated to the solution x is left- and right-continuous at t_c.
(3) The trajectory is transverse to K at $x(t_c)$, i.e. for any $h^\star \in N_K(x(t_c))$ such that there exists $h \in T_K(x(t_c)) \backslash R_K(x(t_c))$ with $h^\star \cdot h = 0$, then one has:
$$h^\star \cdot f(x(t_c), u(t_c)) \neq 0.$$

Hypothesis 5. An optimal trajectory $(x(\cdot), u(\cdot))$ has no $(m = 0)$ or a finite number $m \geq 1$ of regular crossing times $\{t_1, \cdots, t_m\}$ over $[t_0, T]$.

Proposition 6. Suppose that Hypotheses 2 and 5 are fulfilled and let $\mathcal{T}_c := \{t_1, ..., t_m\}$. Then, one has:

(1) $\exists\, p_0 \leq 0$, $p : [t_0, T] \to \mathbb{R}^n$ s.t. $(p_0, p(\cdot)) \neq 0$ and
$$\dot{p}(t) = -\partial_x H(x(t), p(t), p_0, u(t)) \quad a.e.\ t \notin \mathcal{T}_c.$$
(2) $u(t) \in \arg\max_{v \in U} H(x(t), p(t), p_0, v)$ a.e. $t \in [t_0, T]$
(3) Transversality condition : $p(T) = 0$.
(4) The Hamiltonian is constant along the trajectory.
(5) At any crossing time t_c, we have :
$$p(t_c^+) - p(t_c^-) \in N_K(x(t_c)).$$

Moreover $\exists\, h \in N_K(x(t_c))$ with $\|h\| = 1$ s.t. :
$$p(t_c^+) = p(t_c^-) + \frac{p(t_c^-) \cdot (f(x(t_c), u(t_c^-)) - f(x(t_c), u(t_c^+))) + \sigma p_0}{h.f(x(t_c), u(t_c^+))} h$$
where $\sigma = -1$ (inner) or 1 (outer).

Proof. It is based on application of [9, 10, 7].

Recall that a statement of the HMP without the Hypothesis 5 is an open problem.

Another approach to encounter the difficulty due to the discontinuity of the Hamiltonian is to consider a regularization of the criterion.

3. REGULARIZATION OF THE PROBLEM AND CONVERGENCE RESULTS

In this Section we shall assume that K is a convex set (although extensions to prox-regular sets could be achieved and be the matter of a future work). For convex sets, it is natural to consider the Moreau-Yosida approximation (see [2] and references herein for more details on the Moreau envelope). The characteristic function of K^c can be written:
$$\mathbb{1}_{K^c}(x) = \gamma(\chi_K(x)),$$
where $\gamma(v) = 1 - e^{-v}$ and χ_K is the indicator of K :
$$\chi_K(x) := \left| \begin{array}{ll} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{array} \right.$$
Then we recall the following results:
1. When K is convex, the Moreau envelope of K is of class $C^{1,1}$:
$$x \longmapsto e_\varepsilon(x) := \frac{1}{2\varepsilon} d(x, K)^2$$
2. One has $\gamma(e_\varepsilon(x)) \to \mathbb{1}_{K^c}(x)$ when $\varepsilon \downarrow 0$ for any $x \in \mathbb{R}^n$.

Thus we consider the following regularized problem
$$(\mathcal{P}_\varepsilon) : \quad \inf_{u \in \mathcal{U}} J_\varepsilon^T(u) \quad \text{with} \quad J_\varepsilon^T(u) := \int_{t_0}^{T} \gamma(e_\varepsilon(x_u(t)))\, \mathrm{d}t.$$