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A generalized approach to Economic Model Predictive Control with terminal penalty functions

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Abstract: In this paper, we first introduce upper and a lower bounds of the best asymptotic average performance for nonlinear control systems based on the concepts of dissipativity and control storage functions. This allows to extend the formulation and analysis of Economic Model Predictive Control to more general optimal operation regimes, such as periodic solutions. A performance and stability analysis is carried out within this generalized framework. Finally two examples are proposed and discussed to show the merits of the proposed approach.

Keywords: nonlinear predictive control, optimal control, asymptotic stabilization, performance analysis, Lyapunov methods

1. INTRODUCTION

Because of its ability of handling nonlinearity and system constraints, model predictive control (MPC) is becoming increasingly popular in industrial applications and process control, see examples in Qin and Badgwell (2003). This control paradigm normally relegates economic and profitability issues to the design of optimal set-points and suitable pointwise in time constraints. Real-time control is, instead, only concerned with the resulting tracking problem which is translated as an optimization problem over a finite time horizon and with an objective function which is (for the sake of stability) chosen to be positive definite with respect to some equilibrium of interest. In recent years, however, an alternative approach, economic MPC (EMPC), has looked into the issue of directly addressing economic optimization in real time, and to this end, adopts cost functionals which are not required to be positive definite with respect to the equilibrium point.

In this respect, various tools in literature have been proposed and studied in the economic optimization setup. In analogy to Mayne et al. (2000), where three ingredients are elaborated in stabilizing MPC, consisting of terminal cost, terminal constraint and local controller, similar tools have been proposed for Economic MPC and have allowed feasibility, stability and performance analysis of the closed-loop system. In Rawlings et al. (2012), Angeli et al. (2012) and Amrit et al. (2011), asymptotic stability of EMPC with terminal constraints or terminal costs has been proved by using a rotated stage cost in an auxiliary optimization problem, provided that a condition called strict dissipativity is satisfied. Moreover, in these papers, concepts on EMPC are extended to periodic terminal constraint and average constraints. In order to obtain a larger feasibility set, a new generalized terminal state constraint where the terminal state-input pair can be a free variable in optimization process is studied by Fagiano and Teel (2013). Based on the generalized terminal equality constraint, several update rules for the self-tuning terminal weight are illustrated in Müller et al. (2013). Furthermore, in Müller et al. (2014), the closed-loop asymptotic average performance bounds can be improved if the generalized terminal equality is relaxed by regional constraint.

However, optimal regimes of operation may have complex nature, periodic operation can outperform steady-state and even more general regimes of operations could sometimes arise. To deal with such instances, this work will remove terminal equality constraints and employ a suitable notion of "control storage function" (CSF) as the terminal penalty function. The note is organized as follows. Notation and problem setup are described in Section 2. Section 3 provides an estimate to some upper and lower bound for system asymptotic average performance. The extension of EMPC formulation and the closed-loop stability are discussed in Section 4. Two examples indicating the convergence to the best periodic solution are included in Section 5. Section 6 concludes this paper.

2. PRELIMINARIES AND SETUP

2.1 Notation

The Euclidean norm of x is $|x|$. Let symbols $\mathbb{R}$ and $\mathbb{I}$ denote the sets of real numbers and integers, respectively. $\mathbb{I}_{[a,b]}$ denotes the integers $\{a,\ a+1,\ \cdots,\ b\}$ and $\mathbb{I}_{\geq 0}$ denotes the non-negative integers. A continuous function α: $[0,\ \infty) \rightarrow [0,\ \infty)$ is of *class* $\mathcal{K}$, if it is zero at zero and strictly increasing. A continuous function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is positive definite with respect to some point $x_e \in \mathbb{R}^n$ if $\rho(x_e) = 0$ and $\rho(x) > 0$ for all $x \neq x_e$. The distance of a point $x \in \mathbb{R}^n$ to a set Π is denoted as $|x|_\Pi := \min_{z\in\Pi} |x - z|$.

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2.2 Problem setup

We consider finite dimensional discrete-time nonlinear control systems described by difference equations

$$x^+ = f(x, u) \tag{1}$$

with state $x \in \mathbb{X} \subset \mathbb{R}^n$, input $u \in \mathbb{U} \subset \mathbb{R}^m$, and a continuous state transition map $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$. Together with system (1), let us consider a time-invariant, nonlinear, nonconvex, but continuous stage cost given as

$$\ell(x, u) : \mathbb{Z} \to \mathbb{R} \tag{2}$$

where $\mathbb{Z}$ is a compact set capturing the pointwise-in-time state and input constraints which our system is subject to:

$$(x(k), u(k)) \in \mathbb{Z}, \qquad \forall k \in \mathbb{I}_{\geq 0}. \tag{3}$$

Our goal is to enhance profitability by minimizing the economic costs incurred in the long term system operation:

$$V(x, \mathbf{u}) = \sum_k \ell(x(k), u(k)), \quad x^+ = f(x, u), \quad x(0) = x. \tag{4}$$

To this end, we need to identify a viable subset of state space and corresponding control actions. As well known, the notion of control invariant set is crucial in this respect.

Definition 1. A control invariant set is any non-empty closed set $\overline{\mathbb{X}} \subseteq \mathbb{X}$, such that $\forall x \in \overline{\mathbb{X}}, \exists u : f(x, u) \in \overline{\mathbb{X}}$ and $(x, u) \in \mathbb{Z}$. The corresponding input which keeps the system state inside $\overline{\mathbb{X}}$ is denoted as $\overline{\mathbb{U}}(x) := \{u \,|\, (x, u) \in \mathbb{Z} \text{ and } f(x, u) \in \overline{\mathbb{X}}\}$. The set of state and corresponding admissible input pairs is $\overline{\mathbb{Z}} := \bigcup_{x\in\overline{\mathbb{X}}}[\{x\} \times \overline{\mathbb{U}}(x)]$.

Remark 2. We consider the largest control invariant set $\overline{\mathbb{X}} \subseteq \mathbb{X}$. This contains all control invariant sets in $\mathbb{X}$ and any given intial condition $x(0) \notin \overline{\mathbb{X}}$ generates trajectories which violate system constraints (3) at some point in time. Therefore, constraints (3) can be strengthened as follows:

$$(x(k), u(k)) \in \overline{\mathbb{Z}}, \qquad \forall k \in \mathbb{I}_{\geq 0}, \tag{5}$$

and viability is still guaranteed for initial state $x \in \overline{\mathbb{X}}$.

It will also be convenient to define an additional control invariant set for later use as in the assumption below

Assumption 1. There is a control invariant set $\mathbb{X}_f \subseteq \overline{\mathbb{X}}$ and a set of admissible control which keeps the state inside $\mathbb{X}_f$ as $\mathbb{U}_f(x) := \{u \in \overline{\mathbb{U}}(x) \,|\, f(x, u) \in \mathbb{X}_f\}$.

The set of state and corresponding admissible input pairs is denoted as $\mathbb{Z}_f := \bigcup_{x\in\mathbb{X}_f}[\{x\} \times \mathbb{U}_f(x)]$.

3. DISSIPATIVITY AND CONTROL STORAGE FUNCTIONS

In order to have a grasp of the system long-run optimal average performance, three quantities ℓ^*_{av}, $\underline{\ell}$ and $\overline{\ell}$, which are explicitly defined below, will be addressed.

Definition 3. Let $x \in \overline{\mathbb{X}}$ be a given initial state, then the best average asymptotic cost is defined as:

$$\ell^*_{av}(x) := \inf_{\substack{u(\cdot) \\ x(0)=x,\, x^+ = f(x,u) \\ (x(t),u(t)) \in \overline{\mathbb{Z}}}} \liminf_{T\to+\infty} \frac{\sum_{t=0}^{T-1} \ell(x(t), u(t))}{T}. \tag{6}$$

Moreover, we denote by $\ell^*_{av} = \inf_{x\in\overline{\mathbb{X}}} \ell^*_{av}(x)$.

Recall the notion of dissipativity as given in *Definition 4.1* in Angeli et al. (2012),

Definition 4. A discrete time system is dissipative with respect to a supply rate $s : \overline{\mathbb{Z}} \to \mathbb{R}$ if there is a continuous storage function $\lambda : \overline{\mathbb{X}} \to \mathbb{R}$ such that:

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u) \tag{7}$$

for all $(x, u) \in \overline{\mathbb{Z}}$. If in addition a positive definite function $\rho : \overline{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ exists such that:

$$\lambda(f(x, u)) - \lambda(x) \leq -\rho(x) + s(x, u), \tag{8}$$

then the system is said to be strictly dissipative.

Alternatively, given the role of dissipativity in providing lower bounds to the best asymptotic performance, one may consider the following quantity,

Definition 5. The tightest lower bound of ℓ^*_{av} is defined as:

$$\begin{aligned} \underline{\ell} := \sup_c \{\, c \,|\, \exists\, \lambda(\cdot) : \overline{\mathbb{X}} \to \mathbb{R}, \text{continuous, such that} \\ \lambda(f(x, u)) \leq \lambda(x) + \ell(x, u) - c, \forall (x, u) \in \overline{\mathbb{Z}}\}. \end{aligned} \tag{9}$$

Next, along the lines of the well known tool of Control Lyapunov Function (CLF) (see definition in Rawlings and Mayne (2009)), we propose a similar concept referred to as Control Storage Function (CSF).

Definition 6. A control storage function is a function $V_f : \mathbb{X}_f \to \mathbb{R}$ that is continuous and such that for all $x \in \mathbb{X}_f$

$$\inf_{u\in\mathbb{U}_f(x)} V_f(f(x, u)) - s(x, u) \leq V_f(x), \tag{10}$$

where $s : \mathbb{Z}_f \to \mathbb{R}$ is the supply rate.

As a special case of CSF, a CLF, in which $s(x, u) = 0$, is frequently used to approximate the tail of the infinite horizon cost of tracking MPC. Our CSF is meant to be an appropriate choice of terminal cost in an economic setup.

In order to estimate an upper bound for the best asymptotic performance, the quantity below can be specified,

Definition 7. The tightest upper bound of ℓ^*_{av} is defined as:

$$\begin{aligned} \overline{\ell} := \inf_c \{\, c \,|\, \exists\, V_f : \mathbb{X}_f \to \mathbb{R}, \text{ such that } \forall x \in \mathbb{X}_f, \\ \inf_{u\in\mathbb{U}_f(x)} V_f(f(x, u)) + \ell(x, u) \leq V_f(x) + c\}. \end{aligned} \tag{11}$$

Remark 8. Notice that the above CSF inequality in *Definition 6* follows the same form of the Hamilton-Jacobi-Bellman (HJB) inequality, so any CSF can also be regarded as a solution of the HJB inequality.

We are now ready to state the main result of this Section:

Theorem 9. Consider system (1) subject to constraints (5), then, the following inequality holds:

$$\underline{\ell} \leq \ell^*_{av}(x), \quad \forall x \in \overline{\mathbb{X}}. \tag{12}$$

In addition, if *Assumption 1* is fulfilled, we have the following upper bound for $\ell^*_{av}(x)$:

$$\ell^*_{av}(x) \leq \overline{\ell}, \quad \forall x \in \mathbb{X}_f. \tag{13}$$

Proof. The inequality (12) is derived from the dissipativity with supply rate $s(x, u) = \ell(x, u) - \underline{\ell}$ along any feasible solution, whereas (13) is obtained by applying the same technique on CSF with supply rate $s(x, u) = \overline{\ell} - \ell(x, u)$. □

If a system is controllable within finite time to the best optimal operation, every initial condition gives the same