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Parametric reduced order dynamical model construction of a fluid flow control problem

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Abstract: The problem of approximating a given fluid flow control problem, governed by the Navier-Stokes equations at varying Reynolds number, with a low-order parametric linear dynamical model, is presented. To this aim, a three steps approach is proposed: first, *(i)* the original Reynolds number dependent fluid flow problem is spatially and parametrically discretized, then *(ii)* each resulting local very large-scale Linear Time Invariant (LTI) Differential Algebraic Equations (DAE) models are approximated using the **IRKA** approach proposed by Gugercin et al. (2008), and finally *(iii)*, the reduced order models are interpolated and transformed into a low-complexity Linear Fractional Representation (LFR). The overall process is illustrated in a top down framework using a generic flow configuration, namely, an *open cavity* flow. Numerical simulations assess the validity of the approach.

Keywords: Fluid flow model, Navier-Stokes equations, Large-scale model, Model reduction, Linear parametric model.

1. INTRODUCTION AND MOTIVATIONS

1.1 *Fluid flow problem formulation and contribution*

Numerical simulation of dynamical systems plays a pivotal role in many engineering fields to study a wide range of complex physical phenomena (*e.g.* in optimization and control studies). However, the ever-increasing need for accuracy and use of computer-based modelling tools, allowing to well capture the physics, potentially leads to models equipped with extremely large-scale number of variables, equations and state degrees of freedom. Because of the finite machine precision and limited computational burden, the simulation of these models might become numerically inefficient, see *e.g.* (Saad, 2000; Antoulas, 2005). This is the reason why dynamical model reduction (or approximation) offers an interesting remedy to this problem leading to models that are simpler to analyse and faster to simulate while accurately reproducing the original behaviour.

More specifically, within the *fluid flow* community, engineers and researchers are used to manipulate very large-scale DAE models, potentially nonlinear or irrational (*e.g.* obtained from partial differential equations). In their more general form, these dynamical models, might take the following form:

$$\dot{\mathbf{x}}(t) = f\big(\mathbf{x}(t), \mathbf{u}(t), Re\big),\ \ \mathbf{y}(t) = g\big(\mathbf{x}(t), \mathbf{u}(t), Re\big), \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$, $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ and $Re \in \mathbb{R}_+$ are the state-space, input, output vectors and Reynolds number, respectively. Due to their nonlinear behaviour, Reynolds number dependency and large value of n ($\approx 10^7$), these models clearly are inappropriate for any kind of analysis, control and optimization methods [1]. As made clearer in the following sections, we propose to approximate the above initial equation (1) by $\mathbf{H}(Re)$, a low-order DAE *linear parameter dependent model* described as follows:

$$\hat{E}(Re)\dot{\hat{\mathbf{x}}}(t) = \hat{A}(Re)\hat{\mathbf{x}}(t) + \hat{B}(Re)\mathbf{u}(t),\ \hat{\mathbf{y}}(t) = \hat{C}(Re)\hat{\mathbf{x}}(t), \quad (2)$$

where $\hat{E} \in \mathbb{R}^{r\times r}$, $\hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times n_u}$ and $\hat{C} \in \mathbb{R}^{n_y\times r}$ might be linearly Re dependent. The main purpose of transforming (1) into (2) is to be able to construct a simple but representative LFR which is well adapted in the context of robust controller synthesis and analysis methods developed within the control community (Magni, 2006).

In this paper, our objective is intended to bridge the gap between the fluid flow and the control communities by proposing a methodology to simplify any transitional fluid flow configuration problem (evolving at different configurations) by a linear dynamical reduced order parametrized model, appropriate for simulation, analysis and (robust) control [2]. To this aim, a three steps strategy is proposed: *(i)* first, the original flow control problem is linearised around an equilibrium, for varying parametric Reynolds values, then *(ii)* each local very-large scale linear DAE models are approximated using advanced model reduction methods, and finally *(iii)* interpolated to generate a parametric low complexity model, transformed into an LFR.

Throughout this paper, each steps of the proposed methodology is illustrated by means of an interesting flow control configuration known as the *open cavity*, see Figure 1. From a control point of view, as illustrated later on, one of the main interest of this configuration is that, according to the Reynolds number, the model might vary from stable

[1] Indeed, except adjoin methods of Sipp et al. (2010), most of the analysis and control tools are not tailored to this kind of models.

[2] Still, the reader should note that other attempts have already been performed to simplify fluid flow dynamical models using Proper Orthogonal Decomposition, Rational Krylov, Balanced truncation, Balanced POD, Eigen-Realization algorithms etc., as in Barbagallo et al. (2009); Rowley (2005); Ma et al. (2011); Borggaard and Gugercin (2014), but to the authors knowledge, never in the case of low complexity parametrized models.

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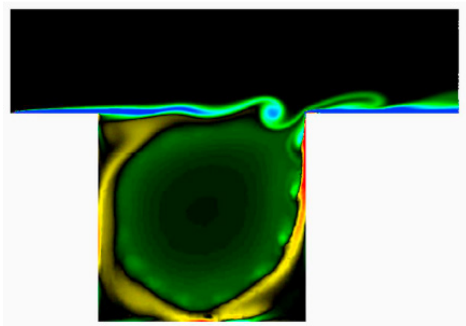

Fig. 1. Open cavity flowfield illustration - from left to right (vorticity snapshot obtained by time-marching the non-linear Navier-Stokes equations).

to unstable which is a challenging issue for both modelling, model reduction and control perspectives. From a fluid point of view, the objective is to maintain the flow laminar in various conditions: in that perspective, linearized models of the Navier-Stokes equations governing the dynamics of small-amplitude perturbations around fixed-points are sufficient to determine a control law from a given sensor.

1.2 Open cavity flow control configuration

The test-case is a two-dimensional open square cavity, which has the same geometry and boundary conditions as that described in Barbagallo et al. (2009). The reference quantities used to non-dimensionalize the governing equations are the uniform flow velocity U_∞, the cavity depth D, the Reynolds number defined as $Re = U_\infty D/\nu$, where ν is the viscosity. The origin of the coordinate system $(x = 0, y = 0)$ is set at the upstream corner of the cavity, so that the downstream edge is at $(x = 1, y = 0)$. Boundary conditions are set as follows: uniform unitary flow at the inlet boundary $(x = -1.2)$, standard exit conditions at the outlet $(x = 2.5)$ [3], free-slip conditions on the upper boundary $(y = 0.5)$ and on the following parts of the lower boundary $(-1.2 \leq x < -0.4, y = 0)$ and $(1.75 < x < 2.5)$, while no-slip conditions are prescribed on the remaining parts of the lower boundary from $(-0.4 \leq x \leq 1.75)$. The mesh used is composed of 193,874 triangles, corresponding to $n = 680,974$ degrees of freedom for the three variables (v_x, v_y, v_z). Here for the spatial discretization of the governing equations, we use Arnold-Brezzi-Fortin MINI-elements with four-node P1b elements for the velocity components and three-node P1 elements for the pressure.

1.3 Paper notations and structure

Throughout the paper, the following notations will be used: $\mathbf{H}$ (*resp.* $H(s)$) denotes the full order state-space model realization (*resp.* transfer function) of order n and $\hat{\mathbf{H}}$ (*resp.* $\hat{H}(s)$) stands for the reduced order state-space model realization (*resp.* transfer function) of order $r \ll n$.

The paper is structured as follows: Section 2 first describes the original open cavity configuration fluid flow nonlinear partial differential equations, then summarizes the linearisation procedure. In Section 3, as rooted on the obtained multiple local very large-scale linear equations, low order DAE models are obtained through an interpolatory method. Then, the interpolation in a state-space realization basis is performed and followed by an LFR generation in Section 4. Conclusions and discussions are then given in Section 5.

[3] We impose $-pn + \nu\nabla v \cdot n = 0$, where n is the outward normal to the domain.

2. LINEARISATION AND DISCRETISATION OF THE OPEN CAVITY FLUID FLOW CONFIGURATION

2.1 Nonlinear infinite dimensional modelling, input and output definitions

The full non-dimensionalized non-linear governing equations are written for the velocity field $\mathbf{v} = (v_x, v_y)$ and the pressure p (Δ is the Laplacian, ∇ the gradient operator):

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \frac{1}{Re}\Delta \mathbf{v} + \mathbf{f}u(t) \ , \ \nabla \cdot \mathbf{v} = 0 \quad (3)$$

where the control input $\mathbf{f}u(t)$ is a volumic forcing on the cross-stream component of the momentum equations located near the upstream edge of the cavity:

$$\mathbf{f} = (0, \exp(-((x + 0.1)^2 + (y - 0.02)^2)/0.014416), 0). \quad (4)$$

A typical snapshot of the flowfield obtained without forcing $(u = 0)$ is shown in Fig. 1. We clearly observe a boundary layer starting at $(x = -0.4, y = 0)$ and a thin shear layer on the cavity lip. This thin shear-layer will generate Kelvin-Helmholtz type instabilities that induce a global instability of the flow in the form of a Hopf bifurcation at $Re = 4140$ (Sipp and Lebedev, 2007). For the output of the control set-up, we consider a shear-stress measurement downstream of the right edge of the cavity:

$$y(t) = \int_{x=1}^{x=1.1} \left.\frac{\partial v_x}{\partial y}\right|_{y=0} \mathrm{d}x. \quad (5)$$

With the actuator $u(t)$ and the sensor $y(t)$, we are therefore led to a single-input-single-output (SISO) sytem.

2.2 Linearisation and discretisation

In the following, we focus on the dynamics of small-amplitude perturbations $(\epsilon \ll 1)$ in the vicinity of a base-flow, which is a fixed point of the Navier-Stokes equations: $\mathbf{x}(t) = \mathbf{x}_0^{(Re)} + \epsilon \mathbf{x}_1(t)$. In the same spirit, we write $y(t) = y_0^{(Re)} + \epsilon y_1(t)$ and $u(t) = \epsilon u_1(t)$. The perturbation $\mathbf{x}_1^{(Re)}(t)$ is then governed by the linearized discretized Navier-Stokes equations around fixed point $(\mathbf{v} = \mathbf{v}_0, u = 0)$:

$$E\dot{\mathbf{x}}_1(t) = A(Re)\mathbf{x}_1(t) + Bu_1(t) \ , \ y_1(t) = C\mathbf{x}_1(t), \quad (6)$$

where $A(Re) = \partial f/\partial \mathbf{x}|_{\mathbf{x}_0^{(Re)}}$ is the dynamical matrix. Then B, C and E are the parameter independent input, output and mass-matrix associated to the finite-element discretization, respectively. C corresponds to the matrix extracting the shear-stress measurement. In the following, to simplify notations, $u_1(t)$ and $y_1(t)$ will be replaced by $u(t)$ and $y(t)$.

Considering $n_s = 4$ frozen Reynolds number values $Re = \{4000, 5250, 6000, 7500\}$, we finally obtain $n_s = 4$ Single Input Single Output (SISO) LTI DAE of order $n = 680,974$, valid around small variations. For the specific considered problem, the reader should note that the Reynolds number dependency appears on the dynamical