ARTICLE IN PRESS

ISA Transactions xxx (2018) 1–6

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

Research article

Input-to-state stability of time-varying nonlinear discrete-time systems via indefinite difference Lyapunov functions

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ARTICLE INFO

Article history:
Received 8 May 2017
Revised 24 January 2018
Accepted 29 March 2018
Available online XXX

Keywords:
Discrete-time system
Time-varying system
Lyapunov function
Input-to-state stability (ISS)
Input-to-state-stable (ISS) Lyapunov function
Indefinite difference Lyapunov function

ABSTRACT

In this paper, we propose new sufficient criteria for input-to-state stability (ISS) of time-varying nonlinear discrete-time systems via indefinite difference Lyapunov functions. The proposed sufficient conditions for ISS of system are more relaxed than for ISS with respect to Lyapunov functions with negative definite difference. We prove system is ISS by two methods. The first way is to prove system is ISS by indefinite difference ISS Lyapunov functions. The second method is to prove system is ISS via introducing an auxiliary system and indefinite difference robust Lyapunov functions. The comparison of the sufficient conditions for ISS obtained via the two methods is discussed. The effectiveness of our results is illustrated by three numerical examples.

1. Introduction

In this paper, we consider a discrete-time system described by

$$x(k+1) = f(k, x(k), u(k)), \tag{1}$$

with the vector field $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $k \in \mathbb{Z}_+$, state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$.

The aim of this paper is to analyse input-to-state stability (ISS) of system (1) utilizing indefinite difference Lyapunov functions.

The concept of ISS for continuous-time systems was officially introduced by Sontag [13] in the late 1980s. The concept defines a stability of property of state trajectories corresponding to initial states and inputs, and implies that bounded inputs lead to bounded outputs. If there is no input to the considered system, the ISS of the system means the origin of the system is asymptotically stable. There are many results of ISS obtained for continuous-time systems, see Refs. [13–16]. The ISS concept is extended to discrete-time systems in Ref. [4]. It is proved in Ref. [4] that the ISS of the discrete-time system is equivalent to the existence of an ISS Lyapunov function. Furthermore, in Ref. [17], ISS with respect to two measurement functions for discrete-time systems is investigated.

Lyapunov function plays an important role in analysing stability of systems, since it does not require the explicit solution of the corresponding ordinary difference (differential) equations. Thus, construction of Lyapunov functions has been an interesting topic. Compared with finding Lyapunov functions for time-invariant systems, construction of Lyapunov functions for time-varying systems is more difficult. In the above papers, the difference (derivative) of a Lyapunov function along the trajectory of the state is supposed to be negative definite. Many researchers are trying to relax this constraint. Some promising results are obtained. In Refs. [1,10,11], stability of an equilibrium of time-varying continuous-time systems is analysed with the help of weak Lyapunov functions with negative semi-definite derivatives or indefinite derivatives. In [2], ISS of time-invariant discrete-time systems is discussed by finite-step Lyapunov functions. Then in [2], ISS of time-invariant discrete-time systems is discussed by finite-step Lyapunov functions. In Ref. [9] the ISS of time-varying continuous-time systems is investigated by weak Lyapunov functions. Moreover, the non-uniform ISS of time-varying continuous-time systems is studied in Ref. [8]. In Refs. [12,18], relaxed conditions on ISS Lyapunov function for time-varying continuous-time systems are discussed. In Refs. [12,18], for $u \equiv 0$, the derivative of Lyapunov function along the trajectory of

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https://doi.org/10.1016/j.isatra.2018.03.022
0019-0578/

Please cite this article in press as: Li H, et al. Input-to-state stability of time-varying nonlinear discrete-time systems via indefinite difference Lyapunov functions, ISA Transactions (2018), https://doi.org/10.1016/j.isatra.2018.03.022

the state is indefinite, that is, the derivative of Lyapunov function along the trajectory of the state may be positive for some periods. In [18], the author generalizes the results of [12]. Inspired by the results of [12,18], we are interested in exploring indefinite difference Lyapunov functions for time-varying discrete-time systems. Discrete-time systems are of interest since they are widely used to study practical phenomena in many application fields such as engineering, chemistry and finance. Furthermore, because the solutions in discrete-time setting are sequence of points rather than continuous functions as in continuous-time setting, it is not possible to derive discrete-time results straightforwardly from their counterparts in continuous-time setting.

In order to demonstrate that the translation of [12] to discrete-time is not entirely straightforward, we briefly describe the key differences between the main results of [12] and this paper. The constraints imposed on our proposed theorems are not the discrete-time version of those on [12, Theorem 2, 3]. In this paper, the proposed sufficient conditions are stronger. The main difficulty of the paper is to figure out the nice conditions (iii of Theorem 2) of the function θ should satisfies. In this paper, the main results are proved in two new ways. We first prove system (1) is ISS by an indefinite difference ISS Lyapunov function. Then we verify system (1) is ISS via introducing an auxiliary system for system (1) and an indefinite difference robust Lyapunov function. The sufficient constraints proposed by the two methods are different. Hence, the results of indefinite difference Lyapunov functions for time-varying discrete-time systems in this paper are worthy of being presented.

The paper is organised as follows: in Section 2, the notations and preliminaries are introduced. In Section 3 we discuss the main results of the paper: we prove that under certain conditions system (1) is ISS via an indefinite difference ISS Lyapunov function (see Theorem 2), and that system (1) is ISS by introducing an auxiliary system and an indefinite difference robust Lyapunov function (see Theorem 3). The comparison of conditions imposed on Theorem 2, Theorem 3 is discussed in Remark 5. Three numerical examples are presented to demonstrate the effectiveness of our results in Section 4. Some concluding remarks are discussed in Section 5.

2. Notations and preliminaries

Let $\mathbb{R}_+$, $\mathbb{Z}_+$ represent the nonnegative real numbers, the nonnegative integers, respectively. For an integer $S \in \mathbb{Z}_+$, we let $[\frac{S}{2}] = \{S_1 \in \mathbb{Z}_+ \,|\, S_1 \geq \frac{S}{2}\}$. The Euclidean norm of the real vector $x \in \mathbb{R}^n$ is $|x|$. The open ball of radius r around z in the norm of $|\cdot|$ is defined by $B(z, r) = \{x \in \mathbb{R}^n \mid |x - z| < r\}$. For a set $\Omega \subset \mathbb{R}^n$, the boundary, the closure and the complement of Ω are denoted by $\partial\Omega$, $\overline{\Omega}$, Ω^C respectively. The *supremum* norm of a function $u : \mathbb{Z}_+ \to \mathbb{R}^m$ is denoted by $|u|_\infty = \sup_{k\in\mathbb{Z}_+} |u(k)|$.

For a constant $R > 0$, the admissible input values are given by $U_R = \overline{B(0, R)} \subset \mathbb{R}^m$ and the admissible value functions by $u \in \mathcal{U}_R = \{u : \mathbb{Z}_+ \to U_R\}$. The kth element of the solution sequence of (1) corresponding to an initial condition x_0 and an input $u \in \mathcal{U}_R$ is denoted by $x(k, x_0, u)$.

We assume the map $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfies $f(k, 0, 0) = 0$ for $k \in \mathbb{R}$.

Let us recall comparison functions which are very useful in stability analysis. We call a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is *positive definite* if it satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$. A positive definite function is of *class* $\mathcal{K}$ if it is strictly increasing and of *class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to *class* $\mathcal{L}$ if $\gamma(r)$ is strictly decreasing to 0 as $r \to \infty$ and a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of *class* $\mathcal{KL}$ if it is of class $\mathcal{K}_\infty$ in the first argument and of class $\mathcal{L}$ in the second argument. For more details about comparison functions, reference [7] is recommended.

We describe the concept of ISS for system (1) which is concerned in this paper.

Definition 1. *System (1) is input-to-state stable (ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for each input $u \in \mathcal{U}_R$ and each $x_0 \in \mathbb{R}^n$, it holds that*

$$|x(k, x_0, u)| \leq \beta(|x_0|, k) + \gamma(|u|_\infty) \tag{2}$$

for each $k \in \mathbb{Z}_+$

Remark 1.

(1) *If $\gamma(\cdot) \equiv 0$ in (2), then the origin of system (1) is robustly asymptotically stable.*
(2) If $u \equiv 0$ *in* Definition 1, *then the origin of system* (1) *without inputs is asymptotically stable.*

Now we introduce the concept of ISS Lyapunov function which can be used to prove system (1) is ISS.

Definition 2. *A continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS Lyapunov function for system (1) if there exist functions α_1, α_2, $\alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that*

$$\alpha_1(|x(k)|) \leq V(k, x(k)) \leq \alpha_2(|x(k)|), \quad \text{for } k \in Z_+,\ x \in \mathbb{R}^n, \tag{3}$$

$$V(k+1, x(k+1)) - V(k, x(k)) \leq -\alpha_3(|x(k)|) + \sigma(|u|_\infty), \tag{4}$$

for $k \in Z_+, x \in \mathbb{R}^n, u \in U_R$.

Considering the following system

$$\begin{cases} t(k+1) = 1 + t(k), \\ x(k+1) = f(k, x(k), u(k)) \end{cases} \tag{5}$$

with the out map $h(t(k), x(k)) = x(k)$. The results of Section 2.3 and Theorem 1 from Ref. [3] lead to the following result.

Theorem 1. *System (1) is ISS if and only if there exists an ISS Lyapunov function.*

Remark 2.

(1) *Using similar technique utilized in [2, Reamrk 3.7], we have the following result. According to* Definition 2, *there exist $\sigma \in \mathcal{K}$ and a positive definite function ρ with $(\mathrm{id} - \rho) \in \mathcal{K}_\infty$ such that*

$$V(k+1, x(k+1)) \leq \rho(V(k, x(k))) + \sigma(|u|_\infty), \tag{6}$$

for $k \in Z_+, x \in \mathbb{R}^n, u \in U_R$.

(2) *If system (1) is ISS, then there exists a continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ is an ISS Lyapunov function, i.e., there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that the inequalities (3) and (4) hold.*
From (4), we have that

$$\begin{aligned} V(k+1, x(k+1)) - V(k, x(k)) &\leq -\alpha_3(|x(k)|) + \sigma(|u|_\infty) \\ &= -\frac{1}{2}\alpha_3(|x(k)|) - \frac{1}{2}\alpha_3(|x(k)|) \\ &\quad + \sigma(|u|_\infty). \end{aligned}$$

Thus if $|x(k)| \geq \alpha_3^{-1}(2\sigma(|u|_\infty))$ holds, then

$$\begin{aligned} V(k+1, x(k+1)) - V(k, x(k)) &\leq -\frac{1}{2}\alpha_3(|x(k)|) \\ &\leq -\frac{1}{2}\alpha_3 \circ \alpha_2^{-1}(V(k, x(k))). \end{aligned}$$

By a standard comparison lemma (see e.g., [6]), there exists a function $\beta \in \mathcal{KL}$ such that

$$V(k, x(k)) \leq \beta(V(x_0), k). \tag{7}$$

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