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Practice article

An efficient algorithm for low-order direct discrete-time implementation of fractional order transfer functions

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ABSTRACT

Fractional order systems become increasingly popular due to their versatility in modelling and control applications across various disciplines. However, the bottleneck in deploying these tools in practice is related to their implementation on real-life systems. Numerical approximations are employed but their complexity no longer match the attractive simplicity of the original fractional order systems. This paper proposes a low-order, computationally stable and efficient method for direct approximation of general order (fractional order) systems in the form of discrete-time rational transfer functions, e.g. processes, controllers. A fair comparison to other direct discretization methods is presented, demonstrating its added value with respect to the state of art.

1. Introduction

Fractional calculus may be generally described as a generalization of integration and differentiation to an arbitrary order [1,2]. Several physical systems have been shown to have constitutive equations of non-integer order [3–6]. The basic element of fractional order systems is the fractional order operator, defined in its continuous form as s^λ, with λ a real number usually chosen in the $(-1\div1)$ range, but not limited. An important property modelled by such systems is that of memory [7]. This property requires a fractional order system of infinite dimension, involving unlimited memory in comparison to the classical integer order systems that are finite dimensional. The challenge for implementing such fractional order systems and controllers is finding their rational approximation [8–10]. Analog realizations of fractional order systems have been presented in Refs. [11,12]. Important features to ensure stability of such equations, in their (non)rational form are discussed in Refs. [13–15].

For a digital implementation of fractional order systems (e.g. controllers) there are two discretization methods: indirect and direct discretization, respectively [16].

In the indirect discretization method, a rational continuous-time approximation is firstly developed, subsequently discretized using any of the well known discretization techniques [17,18]. Among the most widely used continuous-time approximation methods are: the Oustaloup Recursive Approximation method [19], the Carlson method [20] and the Modified Oustaloup Filter [18,21].

An example of a recent indirect discretization method is based on approximating the fractional integrator/differentiator using the CFE expansion approach along with the Al-Alaoui operator [22]. The method presented here seems to be simpler than other methods using directly the discrete-time version of the Grünwald-Letnikov operator. Another approach also based on efficient continued fraction approximation of the fractional order operator is presented in Ref. [23]. The discrete differentiator is expressed as a z-transfer function, whose coefficients are given in closed form

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in terms of the sampling time and an approximation parameter. The method has its limitation, as the application is focused on producing a rational discrete-time approximation for the half order differentiator.

The Laguerre continued fraction expansion of the Tustin fractional discrete-time operator to irreducible Jacobi tri-diagonal matrices is presented in Ref. [24]. The approach is limited to fractional order integrator or differentiators and not to general fractional order systems. Time and frequency domain analysis is performed to study the quality of the approximation. Another recent paper dealing with fractional-order discrete-time linear time-invariant single-input single-output systems is given in Ref. [25]. The approximation is based on a new, two-layer, fractional-order discrete-time Laguerre filters.

Another indirect discretization approach is presented in Ref. [26]. The method is based on using particle swarm optimization (PSO) to approximate fractional order operators and employs an heuristic procedure to optimize the interlacing of zero-pole pairs on the real axis. Once this continuous-time approximation has been optimized, a discretization rule is applied to obtain the discrete approximation. Simulation results are provided to show that the frequency response obtained by PSO improves the approximation offered by other efficient and recent indirect discretization techniques. An efficient implementation of digital non-integer order systems, with applications to controllers for electro-mechanical systems, is presented in Ref. [27]. A consolidated approximation technique is used and practical implementation problems are addressed, such as the effects of the sampling period, of the conversion between analog and digital domain (and vice versa) and the associated quantization.

Direct discretization methods are based on the expansion of a generating function, defined as a mapping relation or formula for conversion from the continuous-time to the discrete-time operator. Most of the research papers dealing with direct discretization methods tackle the problem of approximating the fractional order differentiator/integrator and only very few discuss the performance of the proposed approximation method for more complicated fractional order transfer functions [16,28–30]. Some research papers discuss the discretization of low-pass fractional order filters, such as in Ref. [31].

A different approach in computing the discrete-time approximation of the fractional order integrator or differentiator is proposed in Ref. [52], with extensions to fractional order low-pass filters [31–33]. Their direct discretization method is based on computing first the analytical impulse response of the fractional order system (IRID). Since the analytical computation of the impulse response of the fractional order system is a tedious task, this approach has been solely developed for fractional order integrators/differentiators and first/second order fractional order low-pass filters, which limit the applicability of the method. Another technique has been given in Ref. [34], valid for simple fractional order integrators/differentiators, by keeping the step response invariant, rather than the impulse response. A comprehensive review of numerical tools for fractional calculus and fractional order controls is given in Ref. [35].

Related to the discrete-time approximations of non-rational transfer functions are the methods for identification. Some of these use exogenous inputs such as step response data [36], block pulse functions [37], nonlinear function optimization [38] or combined time-frequency methods [39]. Recent notable methods for identification of generic parametric models use Taylor expansions [40], iterative methods [41–43] or alternative approaches such as multi-innovation theory [44].

In this paper, we propose an original efficient direct approximation method based on the impulse response. The tedious step of computing an analytical form of the impulse response is avoided by using instead the frequency response of a fractional order system. Employing the frequency response as a basis of computing the impulse response allows for increased flexibility of the method: the proposed technique can be applied to any type of fractional order systems to determine its discrete-time approximation.

The paper is structured as follows. Some preliminaries follow this section to allow the reader an overview of the state of art. The proposed direct discretization method is introduced and described in the third section. Several numerical examples showing the effectiveness of the proposed approach in comparison to similar method are described. A conclusion section summarizes the main outcome of this paper.

2. Preliminaries

The most popular generating functions used to map the Laplace operator s to the discrete-time operator z^{-1}, $s^{\pm\alpha} \rightarrow \left(w\left(z^{-1}\right)\right)^{\pm\alpha}$, may be summarized as:

1. Euler rule:

$$w_E\left(z^{-1}\right) = \frac{1 - z^{-1}}{T} \tag{1}$$

2. Tustin rule:

$$w_T\left(z^{-1}\right) = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{2}$$

3. Simpson rule:

$$w_S\left(z^{-1}\right) = \frac{3}{T}\frac{\left(1 - z^{-1}\right)\left(1 + z^{-1}\right)}{1 + 4z^{-1} + z^{-2}} \tag{3}$$

where T stands for the sampling period. Because of the fitting problems associated with Tustin rule, several linear interpolation operators have been proposed, e.g. the Al-Alaoui integral rule:

$$w_A\left(z^{-1}\right) = \alpha w_E\left(z^{-1}\right) + (1-\alpha)\, w_T\left(z^{-1}\right) = \\ = \frac{T(1+\alpha)}{2}\frac{\left(1 + \frac{1-\alpha}{1+\alpha}z^{-1}\right)}{1 - z^{-1}} \tag{4}$$

with $\alpha \in (0\div1)$ a user supplied weight that balances the interpolation between the classical Euler and Tustin rules. For $\alpha = 3/4$, the conventional Al-Alaoui operator is obtained. A similar approach led to a generating function based on a linear combination of Simpson's rule and Trapezoidal integrators [45], i.e. the operator:

$$w_C\left(z^{-1}\right) = k_0\frac{1 - z^{-2}}{\left(1 + bz^{-1}\right)^2} \tag{5}$$

where $k_0 = \left(\frac{6b}{T(3-a)}\right)^{\alpha}$, $b = \frac{3+a-2\sqrt{3a}}{(3-a)}$, with $\alpha \in (0\div1)$ the fractional order of the differentiator and $a \in (0\div1)$, the weighting factor between the Simpson and Tustin rules.

For band-limited rational approximation of fractional order elements, higher order discrete-time transfer functions need to be determined such that they maintain the constant-phase characteristics of the fractional order integrator, within a selected frequency range. To obtain this approximation, several recursive formulae have been considered. The order of the digital filter that approximates the fractional order element is always a compromise between the accuracy and the ease of hardware implementation. In terms of recursive methods, it is well known fact that the PSE (power series expansion) scheme will produce FIR (finite impulse response) filters and requires a higher order of the filter to produce an acceptable accuracy of the approximation [45]. This obviously complicates the analysis and modelling of fractional order systems [17]. CFE (continuous fraction expansion) methods are generally preferred, since they lead to IIR (infinite impulse response) filters, requiring

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