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Research article

Penalty boundary sequential convex programming algorithm for non-convex optimal control problems

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ABSTRACT

A nonlinear optimal control problem with non-convex cost function and non-convex state constraints can be addressed by a series of convex programming to obtain numerical solutions in previous methods. However, a feasible initial solution is essential to ensure the convergence. In this paper, slack variables are added into the model to handle the infeasible initial point and are penalized in the cost. What is more, a new approximation point on the boundary of constraints is embraced in each iteration to increase the similarity to original problem and decrease number of iterations. Thus, a penalty boundary sequential convex programming algorithm is proposed, which is globally convergent to a Karush-Kuhn-Tucker (KKT) point of original problem under mild condition. The theoretical basis is guaranteed by a rigorous proof. Single UAV and multi-robots trajectory planning serve as simulations to verify the validity of the presented algorithm.

1. Introduction

In last fifty years, with the rapid development of space technology and digital computer science, the optimization theory of dynamic systems becomes more and more significant, which forms an important branch of mathematics—optimal control theory [1]. Recently, optimal control theory has a lot of successful applications beyond traditional automatic control, such as space technology, system engineering, economic management, decision making and population control [2]. Numerous approaches have been proposed by researchers to solve optimal control problems effectively. These methods are mainly divided into two categories: indirect methods and direct methods [3]. Indirect methods mainly concern first-order necessary conditions of the optimal control problem, and the conditions are turned into a Hamiltonian Boundary-Value Problem (HBVP). The solution to original problem can be obtained by solving the HBVP [4]. Three most common indirect methods are shooting methods [5], multiple-shooting methods [6], and collocation methods [7]. Direct methods are fundamentally different from indirect methods. In direct methods, the state and/or control of the original optimal control problem are approximated in some appropriate manners, and the optimal control problem is transcribed to a nonlinear programming problem (NLP) [8]. Similar to indirectly methods, direct methods can also be divided into three categories: shooting methods [9], multiple-shooting methods [10], and collocation methods [11].

In recent years, pseudo-spectral methods have increased in popularity [12–14]. Pseudo-spectral method is a global form of orthogonal collocation, i.e., in a pseudo-spectral method, the state is approximated using a global polynomial and collocation is performed at chosen points [3]. Pseudo-spectral approaches have some branches depending on points chosen: Gauss-Lobatto points [15], Legendre-Gauss points [16] and Legendre-Gauss-Radau points [17]. But the main ideas of above methods are same. Both indirect methods and direct methods have their own shortcomings. Convergence radius of indirect methods is small and proper initial points are necessary. While for direct methods, the accuracy of the solution is low and the optimality of the solution need to be guaranteed [3]. The main technology result of this paper can be regarded as a particular kind of direct method, which concerns a non-convex optimal control problem.

Non-convex optimal control problem is a non-convex programming problem in mathematics essentially. In common sense, it is quite difficult to solve a non-convex programming problem because of its nature: there is no bound of operation time and initial guess which should be supplied by a human is required. What is worse, the solution is very likely to be feasible or local optimal rather than global optimal [18]. Therefore, directly solving the optimal control problem by general non-convex programming is not appropriate for onboard applications and how to solve the non-convex optimal control problem stably and quickly

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becomes a primary problem. This problem will be solved perfectly if the non-convex problem can be transformed into a convex problem or a series of convex problems. For convex programming, the local optimal is global optimal, the solution time is bounded and initial guess is not necessary [18]. Unfortunately, most non-convex problems cannot be converted to convex at no expense. In this paper, a kind of nonlinear optimal control problems with non-convex cost function and non-convex state constraints are considered and the corresponding method is presented to reduce the expense.

For non-convex optimal control problems, many researchers have proposed effective methods. Authors present a methodology to non-convex optimal control problems which only involves concave inequality constraints in [19] and its key process is to approximate the concave inequality constraints by successive linearization. On the basis of [19], [20] introduces projection into the algorithm with convergence proof. Above ideas are mainly inspired by Difference of Convex (DC) programming [21]. DC programming is introduced in 1985 by Pham Dinh Tao in the preliminary state, and extensively developed by Le Thi Hoai An and Pham Dinh Tao. Since the proposal of Concave Convex Procedure (CCCP) [22], DC programming is widely applied and becomes increasingly popular.

Our previous work [23] handles a kind of non-convex optimal control problems by sequential convex programming and proposes a globally convergent algorithm with rigorous proof. However, a feasible initial point is essential in the algorithm to ensure the convergence, and the linearization of original model directly chooses the first order Taylor expansion, which is little closer to original model and leads to a low convergence rate. Thus, an exact penalty strategy is introduced to address the infeasible initial case and a boundary point is computed at each iteration to serve as the new approximation point to preserve much more similarity to original problem and reduce the number of iterations. The main contributions of this paper are listed as follows.

- Sequential convex programming method is modified with an exact penalty strategy to handle the infeasible initial condition.
- A new point on the boundary of constraints is computed at each iteration to increase the rapidity of presented algorithm.
- Global convergence of proposed algorithm is rigorously analyzed to guarantee the output of algorithm being the KKT point of original problem.

The paper is organized as follow: Section 2 presents the non-convex optimal control model and sequential convex programming method. Exact penalty model and new approximation point are introduced with the main algorithm in Section 3. The main technical result for corresponding algorithm is analyzed in Section 4. Section 5 provides the simulation results to verify the algorithm, and Section 6 concludes this paper.

2. Problem statement

In this section, the model of the non-convex optimal control problem is presented. Sequential Convex Programming method is introduced as a comparison with the new algorithm.

2.1. Problem model

A nonlinear optimal control problem with non-convex cost and non-convex state constraints in discrete form is addressed in this paper, and denoted as **NCP**.

NCP:

$$\min_{x,u} J = \sum_{i=1}^{N} L\big(x(i), u(i)\big) \tag{1}$$

$$s.\ t.\ \ x(i+1) = f\big(x(i), u(i)\big),\ \ i = 1, \ldots, N-1, \tag{2}$$

$$\| M(i)y(i) + p(i) \| \le q^T(i)y(i) + r(i),\ \ i = 1, \ldots, N, \tag{3}$$

$$s\big(x(i), u(i)\big) \le 0,\ \ i = 1, \ldots, N, \tag{4}$$

$$C(i)y(i) + d(i) = 0,\ \ i = 1, \ldots, N, \tag{5}$$

where Eq. (1) is the cost function, N steps are included in this discrete model. $L(x, u)$ is a cost function of state variable x and control variable u. Eq. (2) is the state equation of x and u, and $f(x, u)$ is the dynamic function of system and can be linear or nonlinear with bounded Hessian. In Eq. (3), $y = (x^T, u^T)^T$ is introduced to describe the problem more concisely. $M(i)$, $p(i)$, $q(i)$, $r(i)$ are matrices in proper dimensions and Eq. (3) is a second-order cone constraint (therefore convex). Without specific instructions, $\|\cdot\|$ means Euclidean norm in remainder of the paper. In Eq. (4), $s(x, u)$ is a function of x and u, and it may be convex, concave, or non-convex but has bounded Hessian, which is the main non-convex part of the model. Eq. (5) is the equality constraint of $x(i)$ and $u(i)$ as well as including initial and final conditions of variables.

The problem can be formulated in a concise optimization form through mathematical transformation. The details can be found in [19]. The new problem is denoted as **P0**.

P0:

$$\begin{aligned} &\min_{y} \quad f(y) \\ &s.\ t. \quad g_i(y) \le 0,\ \ i = 1, \ldots, p, \\ &\qquad\quad h_j(y) = 0,\ \ j = 1, \ldots, q, \end{aligned} \tag{6}$$

where $f(y)$: $\mathbb{R}^n \to \mathbb{R}$, $g_i(y)$: $\mathbb{R}^n \to \mathbb{R}$ and $h_j(y)$: $\mathbb{R}^n \to \mathbb{R}$ are linear or nonlinear and smooth functions on theirs domain with bounded Hessians $\frac{\partial^2 f(y)}{\partial y^2}$, $\frac{\partial^2 g_i(y)}{\partial y^2}$ and $\frac{\partial^2 h_j(y)}{\partial y^2}$, therefore, $f(y)$, $\{g_i(y)\}_{i=1}^{p}$ and $\{h_j(y)\}_{j=1}^{q}$ are twice continuously differentiable.

From Theorem 2 (Convex Concave Procedure) in Appendix B [22], a twice differentiable function with bounded Hessian can be decomposed into a sum of a convex function and a concave function. Hence, $f(y)$, $\{g_i(y)\}_{i=1}^{p}$ and $\{h_j(y)\}_{j=1}^{q}$ are reformulated as

$$\begin{aligned} &f(y) = f_{vex}(y) + f_{cave}(y), \\ &g_i(y) = g_{ivex}(y) + g_{icave}(y),\ \ i = 1, \ldots, p, \\ &h_j(y) = h_{jvex}(y) + h_{jcave}(y),\ \ j = 1, \ldots, q, \end{aligned} \tag{7}$$

where functions with subscript *vex* are strict convex functions with eigenvalues more than zero, and strict concave functions with subscript *cave*. Substituting Eq. (7) into **P0**, the new equality constraint $h_{jvex}(y) + h_{jcave}(y) = 0$ is derived and it can be expressed as two inequality constraints $h_{jvex}(y) + h_{jcave}(y) \le 0$ and