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Research article

# A further result on consensus problems of second-order multi-agent systems with directed graphs, a moving mode and multiple delays☆

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ABSTRACT

This paper considers a consensus problem of a class of second-order multi-agent systems with a moving mode and multiple delays on directed graphs. Using local information, a distributed algorithm is adopted to make all agents reach a consensus while moving together with a constant velocity in the presence of delays. To study the effects of the coexistence of the moving mode and delays on the consensus convergence, a frequency domain approach is employed through analyzing the relationship between the components of the eigenvector associated with the eigenvalue on imaginary axis. Then based on the continuity of the system function, an upper bound for the delays is given to ensure the consensus convergence of the system. A numerical example is included to illustrate the obtained theoretical results.


## 1. Introduction

As a fundamental problem in distributed control of multi-agent systems, consensus problem has received a great deal of attention from the control community [1–24]. The studies on consensus problem mainly concentrate on the communication topologies [1–3], the information delays [4–14], the constraints on the states and inputs [17–20]. In this paper, our objective is to study a consensus problem of a class of second-order multi-agent systems with a moving mode and multiple delays. Current works about consensus problem with delays mainly concentrated on the case where all agents finally converge to a static point. For example, articles [4,5] investigated the consensus problems of first-order multi-agent systems with identical and nonidentical delays, where all agents finally converge to a static point, while articles [6,7] studied the consensus problems of second-order multi-agent systems with nonuniform delays, where all agents also finally reach a consensus at a static point. In some practical applications, agents not only need to reach a consensus but also need to move together with a constant or time-varying velocity. For example, in [1], a consensus problem of second-order multi-agent systems was studied where all agents finally converge to a common point and move together with a constant velocity while in [21–24], distributed circular control problems were studied where all agents finally form a formation and circle around a common point. However, there are rare works concerned about the case where the final consensus point of the agents is moving with a nonzero velocity. Articles [8] and [9] studied the consensus problems for second-order multi-agent systems with time-delay and a moving mode, but the obtained results are limited to either the case where the time-delays are all equal or the case where the graph is undirected.

Founded on the works of [8], we study the consensus problem of a class of second-order multi-agent systems with a moving mode and multiple delays on directed graphs. Using local information, a distributed algorithm is adopted to make all agents reach a consensus while moving together with a constant velocity in the presence of delays. To study the effects of the coexistence of the moving mode and delays on the consensus convergence, a frequency domain approach is employed through analyzing the relationship between the components of the eigenvector associated with the eigenvalue on imaginary axis. Then based on the continuity of the system function, an upper bound for the delays is given to ensure the consensus convergence of the system.

The following notations will be used throughout this paper. $R^m$ denotes the set of all $m$ dimensional real column vector; $\mathbb{C}^m$ denotes the set of all $m$ dimensional complex column vectors; $I_m$ denotes the $m$ dimensional unit matrix; $\otimes$ denotes the kronecker

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product; $(\cdot)^T$ denotes the transpose of vectors or matrixes and $(\cdot)^*$ denotes the conjugate transpose of them. $\mathbf{1}$ represents $[1, 1, \ldots, 1]^T$ with compatible dimensions (sometimes, we use $\mathbf{1}_n$ to denote $\mathbf{1}$ with dimension $n$); $\mathbf{0}$ denotes zero value or zero matrix with appropriate dimensions; $\|\cdot\|$ refers to the standard Euclidean norm of vectors.

## 2. Graph theory and model

Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph, where $\mathcal{V}$ is the set of agents, $\mathcal{E}$ is the set of edges, and $\mathcal{A} = [a_{ik}]$ is a weighted adjacency matrix. The agent indexes belong to a finite index set $\mathcal{I} = \{1, 2, \ldots, n\}$. An edge of $\mathcal{G}$ is denoted by $e_{ik} = (i, k)$, which indicates that agent $i$ can receive information from agent $k$. The adjacency matrix is defined as $a_{ii} = 0$ and $a_{ik} > 0$ when $e_{ik} \in \mathcal{E}$. The set of neighbors of agent $i$ is denoted by $N_i = \{k \in \mathcal{V}: (i, k) \in \mathcal{E}\}$. The Laplacian of the graph $\mathcal{G}$ is defined as $L = [l_{ik}]$, where $l_{ii} = \sum_{k=1}^{n} a_{ik}$ and $l_{ik} = -a_{ik}$, $i \neq k$. A path is a sequence of ordered edges of the form $(i_1, i_2), (i_2, i_3), \cdots$, where $i_k \in \mathcal{V}$. A directed graph is strongly connected if there is a directed path from any node $v_i$ to any other node $v_k$. A directed graph is balanced if $\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{ki}$ for all $i \in \mathcal{I}$. Moreover, a graph is undirected if $a_{ik} = a_{ki}$ for all $i, k$. An undirected graph is connected if there is an edge between every pair of different agents. [25]

**Lemma 1** (*[25]*). *If the graph $\mathcal{G}$ is undirected and connected, then its Laplacian $L$ has properties as follows:*

(1) 0 is a simple eigenvalue with associated eigenvector $\mathbf{1}$
(2) all its other $n-1$ eigenvalues are positive and real.

## 3. Model

Consider a multi-agent system with $n$ agents. Each agent is regarded as a node in a directed graph $\mathcal{G}$. Suppose that each agent has the following dynamics:

$$\begin{aligned} \dot{r}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t) \end{aligned} \tag{1}$$

where $r_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ denote the position and the velocity of agent $i$ for $i = 1, \ldots, n$ and $u_i(t) \in \mathbb{R}$ is the control input. It is assumed that the initial condition $r_i(s)$ and $v_i(s)$ satisfy the dynamics of (1) for all $s \leq 0$.

The goal of the agents is to use local information to reach a consensus and move together with a constant velocity under directed balanced communication graphs, i.e.,

$$\begin{aligned} &\lim_{t\to+\infty} [r_i(t) - r_k(t)] = 0, \\ &\lim_{t\to+\infty} v_i(t) = v_0, \end{aligned} \tag{2}$$

for all $i, k \in \mathcal{I}$ and some constant $v_0$ when multiple delays are involved. To this end, we adopt the following algorithm introduced in [8]:

$$\begin{aligned} u_i(t) = &\sum_{k\in N_i} a_{ik}(r_k(t-\tau_{ik}) - r_i(t-\tau_{ik})) \\ &+ \sum_{k\in N_i} a_{ik}(v_k(t-\tau_{ik}) - v_i(t-\tau_{ik})), \end{aligned} \tag{3}$$

where $\tau_{ik}$ is the communication delay from agent $k$ to $i$. In [8] and [9], consensus of second-order multi-agent systems with a moving mode and delays was studied and conditions were given for the system consensus by finding the explicit relationship between the delays and the eigenvalues of the system matrix. When directed balanced graphs are taken into account, the relationship between the delays and the eigenvalues of the system matrix becomes too complicated to analyze and hence the analysis approaches in [8] and [9] are no longer valid for the system (1) with (3).

For convenience of discussion, let $0 \leq \tau_1 < \tau_2 \cdots < \tau_M$ denote all different delays of the system (1) with (3). Let $\xi(t) = [r_1(t), v_1(t), \ldots, r_n(t), v_n(t)]^T$. Writing the system (1) with (3) in a compact form, we have

$$\dot{\xi}(t) = (I_n \otimes E)\xi(t) - \sum_{m=1}^{M} (L_m \otimes F)\xi(t-\tau_m), \tag{4}$$

where

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

and $L_m$ denotes the component of $L$ corresponding to the delay $\tau_m$ for $m = 1,\ldots,M$. It is clear that $L_m\mathbf{1} = 0$ and $L = \sum_{m=1}^{M} L_m$.

## 4. Main theorem

**Lemma 2** (*[8]*). *Suppose that the directed graph $\mathcal{G}$ is strongly connected and balanced, the matrix $I_n \otimes E - L \otimes F$ has an eigenvalue at 0 with multiplicity 2 and the other eigenvalues have negative real parts.*

Since the graph $\mathcal{G}$ is directed and balanced, we need first study the matrix $\frac{L+L^T}{2}$ before presenting the main results. Since the graph $\mathcal{G}$ is balanced and strongly connected, $\frac{L+L^T}{2}$ can be regarded as the Laplacian of a connected undirected graph. From Lemma 1, all the eigenvalues of $\frac{L+L^T}{2}$ are real and nonnegative. Let $\lambda_{max}$ be the largest eigenvalue of $\frac{L+L^T}{2}$. We have the following theorem.

**Theorem 1.** *Suppose that the directed graph $\mathcal{G}$ is strongly connected and balanced, and $\tau_m = \tau$ for some constant $\tau > 0$ and all $m$. Using (3) for (2),*

$$\lim_{t\to+\infty} [r_i(t) - r_k(t)] = 0,$$

$$\lim_{t\to+\infty} v_i(t) = \sum_{i=1}^{n} v_i(0),$$

*if one of the following two conditions holds:*

(1) $0 \leq \tau < \frac{\arctan\bar{\omega}}{\bar{\omega}}$ where $\bar{\omega} = \sqrt{\frac{\lambda_{max}}{1-\lambda_{max}}}$ when $\lambda_{max} < 1$;.
(2) $0 \leq \tau < \frac{\arctan 2\lambda_{max}}{2\lambda_{max}}$ when $\lambda_{max} \geq 1$.

**Proof.** Consider the frequency transformation of the system (4). It follows that $\xi(s) = \varphi^{-1}(s)\xi(0)$,
where $\varphi(s) = sI_{2n} - I_n \otimes E + \sum_{m=1}^{M} (L_m \otimes F)e^{-s\tau_m}$.

When $s = 0$, $\varphi(s) = -I_n \otimes E + L \otimes F$. According to Lemma 2, it can be obtained that $\varphi(s)$ has an eigenvalue at 0 with multiplicity 2. Moreover, $\varphi(0)(\mathbf{1}_n \otimes [1, 0]^T) = 0$ and $\varphi(0)^2(\mathbf{1}_n \otimes [a, b]^T) = 0$, where $a$ and $b$ are two constants such that $ab \neq 0$. If all other eigenvalues of $\varphi(s)$ have negative real parts, from Lemma 4 in [8], all agents will finally converge to the space spanned by $\mathbf{1}_n \otimes [1, 0]^T$ and $\mathbf{1}_n \otimes [a, b]^T$. That is, consensus can be reached.

To show that all the other eigenvalues have negative real parts, we consider the delays such that $\varphi(s) = 0$ has nonzero roots on the imaginary axis. Suppose that $s = jw \neq 0$ is a root of $\varphi(s) = 0$ on the imaginary axis. Let $q \in \mathbb{C}^{2n}$ be a vector such that $\| q \| = 1$ and

$$\left[ j\omega I_{2n} - I_n \otimes E + \sum_{m=1}^{M} (L_m \otimes F)e^{-j\omega\tau_m} \right] q = 0 \tag{5}$$