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A New Perspective to Synchronization in Networks of Coupled Oscillators: Reverse Engineering and Convex Relaxation

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Abstract: We take a new approach to investigate synchronization in networks of coupled oscillators. We show that the coupled oscillator system when restricted to a proper region is a distributed partial primal-dual gradient algorithm for solving a well-defined convex optimization problem and its dual. We characterize conditions for synchronization solution of the KKT system of the optimization problem, based on which we derive conditions for synchronization equilibrium of the coupled oscillator network. This new approach reduces the hard problem of synchronization of coupled oscillators to a simple problem of verifying synchronization solution of a system of linear equations, and leads to a complete characterization of synchronization condition for the coupled oscillator network in an interesting and practically important region. Our synchronization condition is stated elegantly as the existence of solution for a system of linear equations, of which one best existing synchronization condition is a special sufficient condition case. In addition, we formulate a non-convex optimization problem with the force balance constraint for which the afore convex optimization problem is relaxation, and show that the coupled oscillator system is also a distributed algorithm for solving this non-convex problem. This has interesting implication on exact convex relaxation, and confirms the insight that a physical system usually solves a convex problem even though it may have a non-convex representation.

Keywords: Synchronization, reverse engineering, distributed algorithms, KKT system, convex relaxation, coupled oscillators, networks.

1. INTRODUCTION

The network of coupled oscillators and its synchronization is one of the most investigated network dynamical systems and behaviors. It has broad applications in various disciplines from biology and medicine to chemistry and physics and to engineering and social sciences; see, e.g., Wiesenfeld et al. (1998), Strogatz (2000), Varela et al. (2001), Winfree (2001), Kiss et al. (2002), Strogatz (2003), Jadbabaie et al. (2004), Boccaletti et al. (2006), Paley et al. (2007), Ha et al. (2010), Dorfler and Bullo (2011), Dorfler et al. (2013), Zhao et al. (2014), You and Chen (2014), Li et al. (2014), Mallada and Low (2014). Despite its broad applications, a complete or tight characterization of the condition for synchronization of coupled oscillators is mostly an open question.

In this paper, we consider a general coupled oscillator model that is partly motivated by the frequency dynamics and control in power networks: some of the oscillators are subject to the second-order Newtonian dynamics while the others are subject to the first-order kinematic dynamics, and they are sinusoidally coupled; see, e.g., You and Chen (2014) and Dorfler et al. (2013). This coupled oscillator model and its various special cases have been studied extensively; see, e.g., the above cited references and particularly Dorfler et al. (2013) for a brief review. In particular, Dorfler et al. (2013) presents an elegant closed-form condition for synchronization that significantly improves upon the existing conditions and is provably exact for various interesting network topologies and parameters.

Motivated by our prior work on the reverse engineering of the frequency control in the power network (You and Chen (2014)), we take a new approach to investigate synchronization of coupled oscillators. Specifically, we show that the coupled oscillator system when restricted to a proper region is a distributed partial primal-dual gradient algorithm for solving a well-defined convex optimization problem and its dual. We characterize conditions for synchronization solution of the KKT system of the optimization problem, based on which we derive conditions for synchronization equilibrium of the coupled oscillator network. This new approach reduces the hard problem of synchronization of coupled oscillators to a simple problem of verifying synchronization solution of a system of linear equations, and leads to a complete characterization of synchronization condition for the coupled oscillator network in an interesting and practically important region. Our synchronization condition is stated elegantly as the existence of solution for a system of linear equations, of which one synchronization condition of Dorfler et al. (2013) is a special sufficient condition case.

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We then formulate a non-convex optimization problem with the force balance constraint for which the above-mentioned convex optimization problem is a relaxation. We show that the coupled oscillator system is also a distributed algorithm for solving this non-convex problem. This has an interesting implication on exact convex relaxation: a non-convex problem may be solved through solving its convex relaxation using a carefully chosen algorithm. This kind of exact convex relaxation is a bit different from the conventional one where the optimum of the convex problem is always a feasible point of the original non-convex problem, and confirms the insight that a physical system usually solves a convex problem even though it may have a non-convex representation.

2. SYSTEM MODEL

Consider a network modeled by a connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, with a set $\mathcal{N}$ of nodes and a set $\mathcal{E}$ of undirected links connecting the nodes. Each node $i \in \mathcal{N}$ denotes an oscillator with phase $\theta_i \in \mathbb{R}$ and frequency $\omega_i = \dot{\theta}_i \in \mathbb{R}$, and each link $(i,j) \in \mathcal{E}$ (or $l \in \mathcal{E}$)[1] is associated with a weight or coupling constant $b_{ij} > 0$ (or $b_l > 0$). The node set is partitioned into two disjoint sets $\mathcal{N} = \mathcal{N}^1 \cup \mathcal{N}^2$. Consider the following coupled oscillator system:

$$M_i\dot{\omega}_i + F_i(\omega_i) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} b_{ij}\sin(\theta_i - \theta_j),\ i \in \mathcal{N}^1, \quad (1)$$

$$F_i(\omega_i) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} b_{ij}\sin(\theta_i - \theta_j),\ i \in \mathcal{N}^2, \quad (2)$$

where each oscillator $i \in \mathcal{N}^1$ follows the second-order Newtonian dynamics with an inertia constant $M_i > 0$ and each oscillator $i \in \mathcal{N}^2$ follows the first-order kinematic dynamics. Each oscillator $i \in \mathcal{N}$ is subject to a constant force of $f_i \in \mathbb{R}$ and a frequency-dependent damping of $F_i(\omega_i)$. The function $F_i(\cdot)$ is assumed to be Lipschitz continuous and strictly increasing.

The above coupled oscillator model (1)-(2) is partly motivated by the frequency dynamics and control in the power network, and a huge literature exists on the synchronization of this general system and its various special cases; see, e.g., Dorfler et al. (2013) and You and Chen (2014) and references therein. For instance, for the frequency dynamics of the power network, the set $\mathcal{N}^1$ is the set mechanical generators and $\mathcal{N}^2$ the set of load buses; f_i is the power inject or draw, $F_i(\omega_i) = D_i\omega_i$ with damping coefficient $D_i > 0$,[2] and M_i the generator inertia; and $b_{ij} = \frac{v_i v_j}{x_{ij}}$ with v_i the voltage magnitude at bus i and x_{ij} the reactance of power line (i,j); see, e.g., Bergen and Vittal (2000) and You and Chen (2014).

We aim to characterize conditions under which the network of coupled oscillators has a synchronization equilibrium and its stability.

[1] We use (i,j) and l interchangeably to denote a link in $\mathcal{E}$. Note that in this section (i,j) is an un-ordered pair, i.e., $(i,j) = (j,i)$. But from the next section on, $l \in \mathcal{E}$ is directed and $(i,j) \neq (j,i)$.

[2] Note that this damping term can result from frequency-sensitive load or frequency-based load or generation control. We can include more than one of such terms at each node as in You and Chen (2014), which will not change the structure of the problem and the results of this paper.

Definition 1. (*Synchronization equilibrium*) A synchronization equilibrium $(\omega, \boldsymbol{\theta} = \{\theta_i; i \in \mathcal{N}\}, \boldsymbol{\theta}^0 = \{\theta_i^0; i \in \mathcal{N}\})$ of the coupled oscillator system (1)-(2) is defined by the following relations:

$$\omega_i = \omega,\ i \in \mathcal{N}, \quad (3)$$

$$\theta_i(t) = \theta_i^0 + \omega t,\ i \in \mathcal{N}, \quad (4)$$

$$F_i(\omega) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} b_{ij}\sin(\theta_i - \theta_j),\ i \in \mathcal{N}, \quad (5)$$

where $\theta_i^0 \in [0, 2\pi),\ i \in \mathcal{N}$.

Motivated by the application in the power network where a security constraint $|\theta_i - \theta_j| < \frac{\pi}{2},\ (i,j) \in \mathcal{E}$ is usually imposed (see, e.g., Bergen and Hill (1981), Bergen and Vittal (2000), and Dorfler et al. (2014)), we are particularly interested in the synchronization equilibrium with $|\theta_i^0 - \theta_j^0| < \frac{\pi}{2},\ (i,j) \in \mathcal{E}$.

Definition 2. (*Phase Cohesiveness (Dorfler et al. (2013))*) Given $\gamma \in [0, \frac{\pi}{2})$, a synchronization equilibrium $(\omega, \boldsymbol{\theta}, \boldsymbol{\theta}^0)$ is γ phase cohesive if $|\theta_i^0 - \theta_j^0| \le \gamma,\ (i,j) \in \mathcal{E}$.

2.1 Reverse Engineering of Network Dynamics with Linearized Coupling

Assume that the system is initially at a synchronization equilibrium with a "nominal" frequency ω^n and phases $\theta_i^n,\ i \in \mathcal{N}$ such that $|(\theta_i - \theta_j) - (\theta_i^n - \theta_j^n)| \ll 1,\ (i,j) \in \mathcal{E}$. Let $\tilde{b}_{ij} = b_{ij}\cos(\theta_i^n - \theta_j^n)$, and consider the following system with linearized coupling between oscillators:

$$M_i\dot{\omega}_i + F_i(\omega_i) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} p_{ij},\ i \in \mathcal{N}^1, \quad (6)$$

$$F_i(\omega_i) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} p_{ij},\ i \in \mathcal{N}^2, \quad (7)$$

$$\dot{p}_{ij} = \tilde{b}_{ij}(\omega_i - \omega_j),\ (i,j) \in \mathcal{E}. \quad (8)$$

In the power network application, $b_{ij}\sin(\theta_i - \theta_j)$ is the nonlinear power flow from bus i to bus j, and the above linearization corresponds to the assumption of small phase angle deviation; see, e.g., Bergen and Vittal (2000).

Let $d_i = F_i(\omega_i)$, and $F_i^{-1}(d_i)$ is well-defined because of F_i being strictly monotone. As in You and Chen (2014), we introduce a cost function corresponding to each damping term:

$$C_i(d_i) = \int F_i^{-1}(d_i)dd_i,\ i \in \mathcal{N}, \quad (9)$$

which is a strictly convex function by the assumption on the function F_i, and a convex optimization problem:

$$\min_{\mathbf{d},\mathbf{p}} \quad \sum_{i\in\mathcal{N}} C_i(d_i) \quad (10)$$

$$\text{subject to} \quad f_i = d_i + \sum_{\{j:(i,j)\in\mathcal{E}\}} p_{ij},\ i \in \mathcal{N}, \quad (11)$$

where $\mathbf{d} = \{d_i; i \in \mathcal{N}\}$ and $\mathbf{p} = \{p_{ij}; (i,j) \in \mathcal{E}\}$. The cost function $C_i(d_i)$ and problem (10)-(11) can have different interpretations, depending on specific applications. For instance, in the power network, $d_i = F_i(\omega_i)$ can be the primary frequency control and $C_i(d_i)$ is then the cost associated with the generation control, and problem (10)-(11) is a DC optimal power