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Stability Analysis of Discrete-Time Linear Systems with Unbounded Stochastic Delays

Burak Demirel * **Hamid Reza Feyzmahdavian** *
Euhanna Ghadimi * **Mikael Johansson** *

* *Department of Automatic Control, School of Electrical Engineering, KTH Royal Institute of Technology, Stockholm, SE-10044, SWEDEN (e-mail: {demirel, hamidrez euhanna mikaelj@ee.kth.se).*

Abstract:
This paper investigates the stability of discrete-time linear systems with stochastic delays. We assume that delays are modeled as random variables, which take values in integers with a certain probability. For the scalar case, we provide an analytical bound on the probability to guarantee the stability of linear systems. In the vector case, we derive a linear matrix inequality condition to compute the probability for ensuring the stability of closed-loop systems. As a special case, we also determine the step size of gradient algorithms with stochastic delays in the unconstrained quadratic programming to guarantee convergence to the optimal solution. Numerical examples are provided to show the effectiveness of the proposed analysis techniques.

Keywords: Discrete-time systems; Stochastic delays; Networked control systems; Gradient descent method

1. INTRODUCTION

With the rise of networked control systems, the analysis of dynamical systems with stochastically varying delays has received an increased attention. Communication delays in networked systems evolve in a random fashion due to retransmissions of (randomly) lost packets, the use of random access protocols that wait for the network to become idle to avoid collisions, etc. Guaranteeing the stability of dynamical systems where delays vary stochastically is a challenging task. To tackle the difficulty in the analysis and design, one can consider the worst-case scenario (i.e., the largest delay) but it is not always possible to give a deterministic worst-case bound, and even when an upper bound on the delays can be guaranteed, always considering the worst-case can result in unnecessary conservatism. Therefore, there is a need for new techniques to analyze stability of linear systems with stochastic delays.

Several results are available on stability of linear systems with stochastic delays; see (Antunes et al., 2012, 2013; Demirel et al., 2015, 2013; Gomez et al., 2013; Nilsson, 1998; Xiao et al., 2000). Nilsson (1998) assumed that the communication delay may not grow larger than the sampling interval, and designed an optimal controller without considering any packet losses. In case time delays are larger than one sampling period, Xiao et al. (2000) aimed at designing a set of output feedback controllers for linear systems with random, but bounded, delays that are modeled as finite-state Markov chains. Gomez et al. (2013) considered linear discrete-time systems with bounded stochastically varying delays, and provided a method to investigate the convergence of the mean and second moment of scalar systems. Later, Gomez et al. (2014) proposed necessary and sufficient stability conditions to ensure the mean-square stability of linear discrete-time systems with bounded stochastic delays. In a different line of work, Verriest and Michiels (2009) provided results for several notions of stability of continuous-time linear systems with unbounded stochastic delays.

Research in networked control systems has mainly focused on bounded delays and packet losses. However, it is also possible to observe unbounded delays in networked systems. For instance, Schenato (2009) considered the case where control messages are transmitted over an unreliable communication channel between the controller and the actuator. When the packet sent from the controller to the actuator is lost, the actuator holds the previous control signal. This problem can be represented as a linear system with unbounded random delays. This kind of delays can also appear when multiple plants share the same communication network if a stochastic scheduling protocol is used to orchestrate the medium access and actuators hold their most recently received value in the absence of new data.

In the present paper, we focus on the stability of linear discrete-time systems with stochastically varying, possibly unbounded, delays. These delays are modeled as random variables that increase with probability $(1-p)$ and are reset to zero with probability p. For scalar systems, we provide an analytical condition which guarantees mean-square stability. For higher-order systems, we derive an LMI-based stability test. Finally, we apply the developed techniques to gradient-descent based optimization under stochastic delays and determine the range of step-sizes that guarantee convergence to the optimum.

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2. NOTATIONS AND PRELIMINARIES

2.1 Notations

We write $\mathbb{N}$ for the positive integers and $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ denote the set of real vectors of dimension n. Vectors are written in bold lower case letters and matrices in capital letters. The set of all real symmetric positive semi-definite matrices of dimension n is denoted by $\mathbb{S}^n_{\succeq 0}$. For a square matrix A, $\lambda_1(A)$ denotes its minimum eigenvalue and $\lambda_n(A)$ denotes its maximum eigenvalue in terms of magnitude. The notation$\{x_k\}_{k\in\mathcal{K}}$ stands for $\{x(k) : k \in \mathcal{K}\}$, where $\mathcal{K} \subseteq \mathbb{N}_0$. The probability of an event Ω is denoted by $\mathbf{P}(\Omega)$. When χ is a stochastic variable, $\mathbf{E}[\chi]$ stands for the expectation of χ.

2.2 Preliminaries

Next, we review the key definitions and results necessary for developing the main results of this paper.

We consider the stochastic linear system described by

$$\boldsymbol{x}(k+1) = \mathcal{A}_{\theta(k)}\boldsymbol{x}(k) \ , \quad (\boldsymbol{x}(0), \theta(0)) = (\boldsymbol{x}_0, \theta_0) \ , \tag{1}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state vector, $\mathcal{A}_{\theta(k)}$ is a mode dependent matrix with appropriate dimensions, and $\{\theta(k)\}_{k\in\mathbb{N}_0}$ is an independent and identically distributed random process with probability distribution $\{p_0, p_1, \cdots, p_N\}$. This process represents mode of the system and it takes on values in the discrete space $\{0, 1, \cdots, N\} \subseteq \mathbb{N}_0$.

Definition 1. (**Ji et al. (1991)**). The system (1) is called as mean-square stable if, for every initial state $(\boldsymbol{x}_0, \theta_0)$,

$$\lim_{k\to\infty} \mathbf{E}\Big[\boldsymbol{x}^\intercal(k)\boldsymbol{x}(k)\Big] = 0 \ .$$

Theorem 2. (**Costa et al. (2005)**). The system (1) is mean-square stable if and only if there exists a positive-definite matrix $X \in \mathbb{S}^n_{\succ 0}$ that satisfies

$$X - \sum_{i=0}^{N} p_i \mathcal{A}_i^\intercal X \mathcal{A}_i > 0 \ , \tag{2}$$

for all $i \in \{0, 1, \cdots, N\}$.

It is worth noting that the result in Theorem 2 can be extended for countably infinite number of systems (i.e., $N \to \infty$) in essence of geometric distribution. Due to the page limitation, we omit this extension.

Definition 3. Suppose that $A \in \mathbb{R}^{n\times n}$. Then, the sequence $\{S_n\}_{n\in\mathbb{N}_0}$ defined by

$$S_n \triangleq \mathbf{I} + A + A^2 + \cdots + A^{n-1} \ , \tag{3}$$

is called the *geometric series* generated by A.

Theorem 4. The geometric series generated by A converges if and only if $\lambda_i(A) < 1$ for each eigenvalue of A. If this condition holds, then $(\mathbf{I} - A)$ is invertible, and

$$S_n \triangleq \sum_{i=0}^{n-1} A^i = (\mathbf{I} - A)^{-1}(\mathbf{I} - A^n) \ , \tag{4}$$

converges to

$$\sum_{i=0}^{\infty} A^i = (\mathbf{I} - A)^{-1} \ . \tag{5}$$

Lemma 5. The discrete-time algebraic Lyapunov equation

$$X - pA^\intercal X A = Q \ , \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$ and $X, Q \in \mathbb{S}^n_{\succeq 0}$, has a unique solution if and only if $p\lambda_n(A)^2 < 1$. The unique solution of (6) can be expressed as an infinite series:

$$X = \sum_{k=0}^{\infty} p^k (A^\intercal)^k Q A^k \ . \tag{7}$$

3. PROBLEM FORMULATION

Consider the following linear dynamical system with stochastic time-varying delays

$$\mathcal{G}: \quad \boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{x}\big(k - d(k)\big). \tag{8}$$

Here, $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, and $A, B \in \mathbb{R}^{n\times n}$ are system matrices. The time delay in system $\mathcal{G}$ is modeled by a sequence $\{d(k)\}_{k\in\mathbb{N}_0}$ of random variables taking their values in $\mathbb{N}_0$, and it evolves according to:

$$d(k+1) = \begin{cases} 0, & \text{with probability } p, \\ d(k)+1, & \text{with probability } 1-p, \end{cases} \tag{9}$$

with $d(0) = 0$.

Let us define $\{l(t)\}_{t\in\mathbb{N}_0}$ as a sequence of increasing integers keep tracking the update moments, i.e.,

$$l(t) \triangleq \min\{k \geq l(t-1) + 1 : d(k) = 0, k \in \mathbb{N}_0\},$$

with $l(0) = 0$. Moreover, for $t \in \mathbb{N}_0$, we define $\boldsymbol{z}(t) \triangleq \boldsymbol{x}(l(t))$ and $\theta(t) \triangleq d(l(t) - 1)$. Now, by combining (8) and (9), system $\mathcal{G}$ can be described by the following jump linear system:

$$\mathcal{G}': \quad \boldsymbol{z}(t+1) = \mathcal{A}_{\theta(t)}\boldsymbol{z}(t), \quad \boldsymbol{z}(0) = \boldsymbol{x}_0 \ , \tag{10}$$

where

$$\mathcal{A}_i = A^{i+1} + \sum_{j=0}^{i} A^j B,$$

when $\theta(t) = i$. The probability having $\mathcal{A}_i$ can be obtained as $\mathbf{P}(\theta(t) = i) \triangleq p(1-p)^i$.

Our goal is to study the mean-square stability of stochastic linear systems of the form (8) and (9).

4. STABILITY ANALYSIS IN CONTROL SYSTEMS

4.1 First-order systems

For pedagogical ease, we first restrict our attention to the case when $\mathcal{G}$ is a first-order system (i.e., $n = 1$). To emphasize that A and B are scalars, we write $A = a$ and $B = b$.

Theorem 6. Suppose that $|a| > 1$ and $|a + b| < 1$. The system (8) with random delays, governed by (9), is mean-square stable if

$$p > \frac{(a+1)(a-b-1)}{a - ab + a^2}. \tag{11}$$

Theorem 6 provides a sufficient condition for the mean-square stability of first-order systems of the form (8) and (9). Since $|a + b| < 1$, the delay-free system ($p = 1$ and, hence, $d(k) = 0$) is asymptotically stable. On the