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Capacitance and forces for thick circular electrodes

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ABSTRACT

Some new results are presented concerning the forces between two equal circular electrodes of finite thickness. For close electrodes different scenarios result, depending on the thickness and on the ratio of charges of the conductors. Attractive or repulsive forces can appear depending on the parameters and on the separation of the electrodes. The force between the electrodes can be non monotonic as a function of the distance and one or more equilibrium points can appear. A unified description of cylindrical systems using a high precision method based on a Galerkin expansion is given and we check our results with a quite accurate boundary element method (BEM).

1. Introduction

The problem of the two discs capacitor has a long tradition in the physical and mathematical literature. In particular, the behaviour of capacitance for close approach of the plates has been studied by Kirchhoff in his pioneering work [1] and the results for the case of flat discs have been brought to a higher degree of approximation in Refs. [2–5]. The formulation of the problem in terms of a dual integral equation in Refs. [6–9] allowed a powerful analytical and numerical framework for the problem and permitted a rigorous derivation [10] of Kirchhoff's result for flat discs (see eq. (10) below). The logarithmic divergence in the subleading term of Kirchhoff formula is produced by an excess of charge density $\Delta\sigma \sim 1/x$, near the edge, where x is the distance from the edge. This is a general behaviour for a capacitor composed by two flat, parallel and equal conductors. The overall excess charge is obtained by integration of $\Delta\sigma$ along the contour, giving the logarithmic correction (see p. 18 of [11]). A generalization of these results in the case of flat discs of different radii has been presented in Refs. [12,13]. The computation of capacitance coefficients is clearly of some interest in itself. In this respect, the main result obtained in this work is a systematic correction to the Kirchhoff approximation [1,14] for the mutual capacitance C at short distances when thick discs are considered.

From a different point of view, the study of the forces between conductors has received new attention after the work [15] where it was found that, in general, two spheres at close approach attract each other even if bringing charges of the same sign. A similarly surprising, but opposite, result emerges for planar electrodes for which a repulsive force can occur when the distance between the plates is very short even if the plates bring charges of opposite sign. In Refs. [13,16,17] it is discussed how this different behaviour depends on the difference between the quadrupole-like charge distribution present for discs and the polarizability effect, dominant in the case of spheres, and absent for flat plates. For definiteness, let us consider the case of two flat discs with positive and different charges $Q_1 > Q_2$. Disc 1 push a positive charge toward the edge of disc 2 creating a negative density in the bulk overlapped region. The interior part of the discs tend to attract each other and the edges, likely charged, repel. The prevailing contribute depend on the geometry and on the ratio Q_1/Q_2 and, consequently, the features of the force at close approach is not generally predictable a priori. However, our general analysis allows us to state that for equal electrodes with a planar contact, there is always a finite gap around the ratio $Q_1/Q_2 = 1$ in which the force is repulsive. This statement can be easily generalized to different electrodes with a planar contact zone, see the discussion after equation (32). A limit case is obtained for two equal flat conductors, where the repulsive force due to edges is logarithmically divergent overcoming anyway the attractive finite force due to the interior except the particular case $Q_1 = -Q_2$. This effect is rigorously proved for equal flat disks in Refs. [13,17] and extended numerically to equal square electrodes in Ref. [16]. For realistic capacitors new scales enter into the problem, in particular for circular electrodes we can have the difference between the radii and the thickness of the material. It is expected that these scales provide a cutoff to the logarithmic divergence leading back to the general scenario sketched above. This expectation has been confirmed analytically and numerically in Ref. [13] in the case of two flat disks of different radius, and is expected to hold also for thick electrodes. A major physical motivation of this work is the confirmation of this hypothesis.

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The problem of forces between two close conductors has an obvious importance in the physics of nano-electromechanical systems (NEMS). The measure of subtle effects as Casimir forces, requires a control on various effects, the main being the elastic stresses and the electrostatic forces, see for example [18,19].

The numerical results are obtained by an application of the Galerkin method, and checked by an accurate version of Boundary Element Method (BEM), where both source and field patches are integrated. Almost all the matrices needed in the Galerkin method were calculated analytically allowing to obtain a high precision in the results. For shortness we omit the mathematical details and the applications to other interesting problems, these topics are covered in Ref. [20].

The paper is organized as follows. In section 2 we give a short discussion of the physics of the system and review briefly the mathematical techniques used in this work. In section 3 we present our results for the capacity matrix for different values of thickness. In this section we discuss on a small deviation from Kirchhoff approximation. In section 4 we apply the previous results to the study of forces, constructing a "phase diagram" which allows to identify for which values of thickness and ratio of charges, the two electrodes attract or repel at short distance. Some examples of the relation force vs distance are shown, exhibiting some possible behaviours. In section 5 we summarize our results and point out some possible extensions.

2. Physical discussion of the system

In a system of conductors the charges Q_i and the potentials V_i are related by

$$Q_i = \sum_j C_{ij} V_j \; ; \qquad V_i = \sum_j M_{ij} Q_j \tag{1}$$

where M_{ij}, the potential matrix, is the inverse of the symmetric capacitance matrix C_{ij}. In terms of these matrices the energy takes the form

$$W = \frac{1}{2}\sum_{ij} C_{ij} V_i V_j = \frac{1}{2}\sum_{ij} M_{ij} Q_i Q_j \tag{2}$$

In the following, we consider two equal and parallel discs of radius a and thickness b (see Fig. 1). The symmetry of the problem implies $C_{11} = C_{22}$ (and $M_{11} = M_{22}$). The distance between the nearest surfaces of the two electrodes will be denoted by ℓ. a is our unit of length and we use dimensionless variables $\kappa = \ell/a$, $\tau \equiv 2h = b/a$. For $\tau = 0$ the conductors degenerate in two flat discs. Having to compare different configurations of the system, the functional dependences on τ and κ will be written explicitly as $C_{ij}[\tau, \kappa]$, when necessary.

In Refs. [12,13,16,17] it has been pointed out that the capacitance coefficients can be organized in a hierarchy depending on their behaviour for $\kappa \to 0$. In particular, the sum $C_{11} + C_{12}$ stays finite in the limit $\kappa \to 0$. Therefore, it is useful to consider as independent quantities

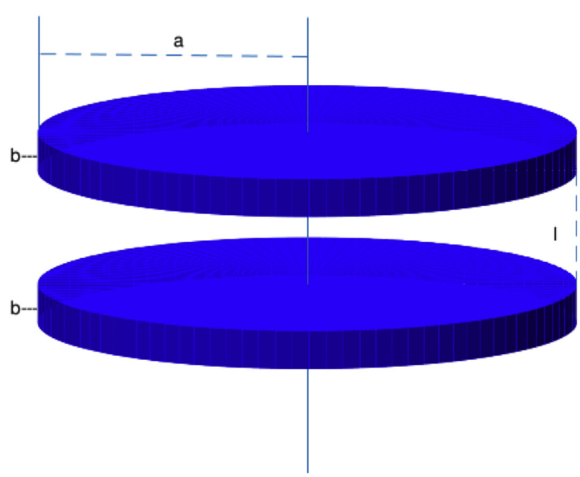

Fig. 1. Sketch of the discs. Thickness of each disc: $b = \tau a$. Distance between discs.$\ell = \kappa a$.

$$C = \frac{C_{11} - C_{12}}{2} \; ; \qquad C_{g_1} = C_{11} + C_{12} \tag{3}$$

C is the usual mutual capacity, i.e. the quotient between the charge on one electrode and the potential difference when the two electrode are held at opposite charges. The total sum C_g of the matrix elements C_{ij} tends, in the limit $\kappa \to 0$, to the capacity of the conductor obtained by the "fusion" of the two elements: in the present case a cylinder of thickness 2τ. Using the notation C_1 for this capacity, the above statement reads

$$C_g[\tau, \kappa] = C_{11} + C_{12} + C_{21} + C_{22} \equiv 2C_{g_1}[\tau, \kappa] \xrightarrow[\kappa \to 0]{} C_1[2\tau] \tag{4}$$

In terms of the above quantities the force between the two electrodes can be written [16]

$$F = -\frac{\partial}{\partial \ell} W = \frac{1}{4}\frac{(Q_1 + Q_2)^2}{C_{g_1}^2}\frac{\partial}{\partial \ell} C_{g_1} + \frac{1}{8}\frac{(Q_1 - Q_2)^2}{C^2}\frac{\partial}{\partial \ell} C \, . \tag{5}$$

In our works mentioned above it is shown that the second term in (5) produces a constant attractive force at short distances, for planar contacts of flat electrodes. The first term is in general repulsive, and in the particular case of two equal discs it is logarithmically divergent [13,17] for $\kappa \to 0$. In the following we will show how this behaviour is smoothed by the thickness.

The results for planar discs ($\tau = 0$) follow from the analytical known behaviour for $\kappa \to 0$:

$$\begin{aligned} C[0, \kappa] \to\ & a\left\{\frac{1}{4\kappa} + \frac{1}{4\pi}\left[\log\left(16\pi\frac{1}{\kappa}\right) - 1\right]\right\} \\ & + a\left\{\frac{1}{16\pi^2}\kappa\left[\left(\log\frac{\kappa}{16\pi}\right)^2 - 2\right]\right\}, \end{aligned} \tag{6}$$

and

$$C_{g_1}[0, \kappa] \to a\left[\frac{1}{\pi} + \frac{\kappa}{2\pi^2}\left(1 - \log\left(\frac{\kappa}{\pi}\right)\right)\right]. \tag{7}$$

A summary of these results and the comparison with numerical computations is given in Ref. [13].

For the case $\tau \neq 0$, to our best knowledge, the only known result is the original computation of Kirchhoff [1,14], confirmed in Ref. [2]. Here and in the following the quantities referred to Kirchhoff's calculation are denoted with the subscript K.

$$\begin{aligned} C_K[\tau, \kappa] \to\ & a\left\{\frac{1}{4\kappa} - \frac{1}{4\pi}\left[1 + \log\frac{\kappa}{16\pi} - \left(1 + \frac{\tau}{\kappa}\right)\log\left(1 + \frac{\tau}{\kappa}\right)\right.\right. \\ & \left.\left. + \frac{\tau}{\kappa}\log\frac{\tau}{\kappa}\right]\right\} \equiv a\, f_K(\kappa, \tau) \end{aligned} \tag{8}$$

while for C_{g_1} we have, from (4)

$$C_{g_1}[\tau, \kappa] \xrightarrow[\kappa \to 0]{} \frac{1}{2} C_1[2\tau] \, . \tag{9}$$

Formula (8) exhibits clearly the problem introduced by the thickness. In the region $b \ll \ell \ll a$, i.e. $\tau \ll \kappa \ll 1$, the system is to all effects equivalent to a capacitor composed by two planar discs, but for $\kappa \ll \tau \ll 1$ the approach for $\kappa \to 0$ changes. From (8) for $\tau \to 0$

$$C_K[\tau, \kappa] \to C[0, \kappa] \sim a\left\{\frac{1}{4\kappa} - \frac{1}{4\pi}\left(1 + \log\frac{\kappa}{16\pi}\right)\right\} \tag{10}$$

in agreement with (6). In the limit $\kappa \to 0$ at fixed τ we have instead, from equation (8)

$$C_K[\tau, \kappa] \to a\left\{\frac{1}{4\kappa} - \frac{1}{2\pi}\log\kappa + \frac{1}{4\pi}\log\frac{\tau}{16\pi}\right\} \tag{11}$$