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Capacitance and potential coefficients at large distances

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ABSTRACT

The design of capacitive sensors and devices for new and emerging applications would benefit from simple and reliable methods to estimate the capacitance between conductors in terms of the capacitance of the isolated bodies and of the distance between them. The coefficients of potential and capacitance of a pair of conductor are approximated with the first terms of an expansion formula in the inverse of their distance. The form given applies to conductors of generic shapes and position in space. A comparison with the exact value for two spheres shows agreement even for rather small distances.

1. Introduction

It is well-known that a formal calculation of capacitance coefficients of a system of conductors can be achieved through the Green's function method [1], but closed and exact analytic expressions are possible only for certain symmetries. Moreover, numerical algorithms [2] are generally cumbersome ad time-consuming, being bound, for each single calculation, to an individual choice of parameters. On the other hand, the wide spread of capacitive sensors and devices makes it interesting to have simple and roughly reliable methods to estimate the capacitance of a couple of conductors in terms of the capacitance of the isolated bodies and of the distance among the conductors. Particularly, it could be useful to dispose of analytical forms describing accurate asymptotic behaviours when a quite wide separation among electrodes with respect to their dimensions occurs. The application of the Green's function method in conjunction with a perturbative approach gives the first terms of an expansion of the coefficient of capacitance for a generic pair of conductors in power of the inverse of the distance of their centres of charge, having as parameters the electrical polarizability, the electric quadrupole coefficient and the intrinsic capacitance of each of the two bodies.

Asymptotic formulas for coefficients of capacitance and potential in particular cases were given since the unprecedented work of J.C.Maxwell [3] both in the near limit and in the far limit regimes, i.e. when the two conducting bodies are very near or very far each other. Specifically, in Maxwell's Treatise the case of two spherical conductors is considered and the exact solution has been written expanding it in power of the distance between the center of the spheres providing manageable formulas which differ in the regions of validity less than one per cent from the exact solution. Even recently, some works were devoted to a critical analysis of approximate formulas in the two regimes [4] [5], for the case of spheres. Valuable would be an approximate formula for the coefficients of capacitance and potentials in the far limit with no restriction on the shape and the orientation of the conductors. The results given has general validity and its comparison with cases in which the coefficients are known analytically or numerically are satisfactory. The calculus appears also useful for applications as the precise knowledge of the asymptotic form in the far limit together with the discrepancy of the measured value should indicates the presence and the magnitude of external disturbances in the system.

The paper is organized as follows. Section 2.1 provides the derivation of the result based on a physical description of the system. The result is confirmed in a more mathematical way in Section 2.2. Section 3 contains a comparison with some analytical results and a brief discussion on the accuracy of the result with respect to possible experimental measures.

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2. Method

2.1. Physical derivation of the result

The charges Q_i on the surfaces of a system of conductors ($i = 1,2,\ldots,N$) at equilibrium are a linear superposition of the uniform potentials V_i of the bodies:

$$Q_i = C_{ij}V_j; \qquad V_i = M_{ij}Q_j. \tag{1}$$

Repeated indices are summed. The coefficients C_{ij} form the capacitance matrix C and depend on the geometry of the problem (shapes, distances etc.). The matrix $M = C^{-1}$ is the potential matrix. C and M are symmetric matrices.

Our goal is to give a simple expression for these matrices in the case of a system of two conductors, a,b, at large distance R, specified below, compared to their sizes, as a function of few *intrinsic parameters* of the separated bodies. The leading order in the expansion is given in textbooks [6]:

$$M = \begin{pmatrix} \frac{1}{C_a} & \frac{1}{R} \\ \frac{1}{R} & \frac{1}{C_b} \end{pmatrix}; \qquad C = \begin{pmatrix} C_a\left(1+\frac{C_aC_b}{R^2}\right) & -\frac{C_aC_b}{R} \\ -\frac{C_aC_b}{R} & C_b\left(1+\frac{C_aC_b}{R^2}\right) \end{pmatrix}. \tag{2}$$

The computation will be extended to the next order, $1/R^3$ for C_{ab} and $1/R^4$ for diagonal elements.

The simplest starting point is to consider the energy of the system expressed in terms of the matrix M

$$U = \frac{1}{2}M_{aa}Q_a^2 + M_{ab}Q_aQ_b + \frac{1}{2}M_{bb}Q_b^2. \tag{3}$$

Up to order $1/R^4$ this energy can be written as the sum of three contributions: (a) a self-energy term, (b) an interaction term, and (c) an induced dipole term. In order to define uniquely the coefficients of the expansion it is necessary to specify more precisely the distance R. Our choice is to define R as the distance between the centres of charge of the *isolated* bodies. This choice is particularly convenient as the intrinsic dipole vanishes and the form of the coefficients results simpler (see also Section 2.2). For those reasons, it is maintained at each step of the computation.

a) A self energy term:

$$U_{self} = \frac{1}{2}\frac{Q_a^2}{C_a} + \frac{1}{2}\frac{Q_b^2}{C_b} \tag{4}$$

C_a,C_b are the self-capacitance of the conductors, and are supposed known.

b) The interaction energy for fixed charge distribution, i.e. for the distribution of charges in equilibrium for *isolated* conductors. With the convention for R specified above, the interaction energy is

$$U_{int} = Q_b\Phi_a(b) + \frac{1}{6}Q_bD_{ij}^{(b)}\partial_i\partial_j\Phi_a(b) + Q_a\Phi_b(a) + \frac{1}{6}Q_aD_{ij}^{(a)}\partial_i\partial_j\Phi_b(a). \tag{5}$$

$D_{ij}^{(a,b)}$ denotes the quadrupole for unit charge of the conductors and $\Phi_a(b)$ is the value of the potential generated by the conductor a at the center of the conductor b. The same convention is used for derivatives and similarly for $\Phi_b(a)$. For our needs is sufficient to take only the first few terms of the multipole expansion

$$\Phi_a(b) = \frac{Q_a}{R} + \frac{1}{2}\frac{1}{R^3}Q_aD_{ij}^{(a)}n_in_j \tag{6}$$

and the similar expression for $\Phi_b(a)$. $\mathbf{n} = \mathbf{R}/R$ is the unit vector between the two centres. Let us note that our choice for R has excluded the dipole interaction term in (5) and consequently, the term proportional to $1/R^2$ in (6).

Substituting (6) in (5) and taking the derivatives an easy computation gives

$$U_{int} = Q_aQ_b\left(\frac{1}{R} + \frac{1}{R^3}\,\frac{1}{2}\left(D_{ij}^{(a)} + D_{ij}^{(b)}\right)n_in_j\right). \tag{7}$$

c) A third term is due to induced dipoles in the conductors:

$$U_{dip} = -\frac{1}{2}\alpha_{ij}^{(a)}E_i^{(b)}(a)E_j^{(b)}(a) + (a \leftrightarrow b) \tag{8}$$

E is the electric field and the notation is the same as for the potentials. It is worth to note that even if the dipole of the isolated body vanishes, an external electric field at the chosen coordinate origin gives rise to an induced dipole. In (8) $E_j^{(a)}(b)$ is the j-th cartesian component of the electric field generated by conductor a at points of the conductor b. In our approximation, it is sufficient to take only the leading coulombic term $\mathbf{E} = Q\mathbf{n}/R^2$ and substituting in (8) we have

$$U_{dip} = -\frac{1}{2}\frac{1}{R^4}n_in_j\left(Q_a^2\alpha_{ij}^{(a)} + Q_b^2\alpha_{ij}^{(b)}\right). \tag{9}$$

The sum of the three contributions (4, 7, 9) has the form (3) with the matrix M given by

$$M = \begin{pmatrix} \frac{1}{C_a} - \frac{\alpha^{(b)}}{R^4} & \frac{1}{R} + \frac{1}{2}\frac{D}{R^3} \\ \frac{1}{R} + \frac{1}{2}\frac{D}{R^3} & \frac{1}{C_b} - \frac{\alpha^{(a)}}{R^4} \end{pmatrix}. \tag{10}$$

with the notation

$$\alpha^{(I)} = \alpha_{ij}^{(I)}n_in_j \qquad D = n_in_j\left(D_{ij}^{(a)} + D_{ij}^{(b)}\right). \tag{11}$$

Inversion of (10) gives to order $1/R^4$

$$\begin{aligned} C_{aa} &= C_a\left(1 + \frac{C_aC_b}{R^2} + \frac{1}{R^4}\left(C_a^2C_b^2 + C_aC_bD + C_a\alpha^{(b)}\right)\right) \\ C_{bb} &= C_b\left(1 + \frac{C_aC_b}{R^2} + \frac{1}{R^4}\left(C_a^2C_b^2 + C_aC_bD + C_b\alpha^{(a)}\right)\right) \\ C_{ab} &= C_{ba} = -\frac{C_aC_b}{R}\left(1 + \frac{D}{2R^2} + \frac{C_aC_b}{R^2}\right). \end{aligned} \tag{12}$$

The first order in this expansion in given in any textbook, see e.g.