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Distributed Coordinated Control of Large-Scale Nonlinear Networks ⋆

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Abstract:
We provide a distributed coordinated approach to the stability analysis and control design of large-scale nonlinear dynamical systems by using a vector Lyapunov functions approach. In this formulation the large-scale system is decomposed into a network of interacting subsystems and the stability of the system is analyzed through a comparison system. However finding such comparison system is not trivial. In this work, we propose a sum-of-squares based completely decentralized approach for computing the comparison systems for networks of nonlinear systems. Moreover, based on the comparison systems, we introduce a distributed optimal control strategy in which the individual subsystems (agents) coordinate with their immediate neighbors to design local control policies that can exponentially stabilize the full system under initial disturbances. We illustrate the control algorithm on a network of interacting Van der Pol systems.

Keywords: Vector Lyapunov functions, comparison equations, sum-of-squares methods.

1. INTRODUCTION

Distributed coordinated control has recently provided powerful control solutions when the conventional centralized methods fail due to inevitable communication constraints and limited computational capabilities. Paradigmatic examples are provided by cooperative and coordinated control for autonomous multi-agent systems (see Bullo et al. (2009)) or large scale interconnected systems (see Zečević and Šiljak (2010)). Distributed coordinated control uses *local* communications between agents to achieve *global* objectives that reflect the desired behavior of the multi-agent system. Usually, a two-level hierarchical multi-agent system is employed, which consists of upper level agent for implementing coordinated control and lower level agents for implementing decentralized control. In this paper, we propose to use this conceptual framework to design distributed coordinated control of large scale interconnected system using vector Lyapunov functions (see Bellman (1962); Bailey (1966)) and comparison principles (see Brauer (1961); Beckenbach and Bellman (1961)). The formulations using vector Lyapunov functions are computationally very attractive because of their parallel structure and scalability. However computing these comparison equations, for a given interconnected system, still remained a challenge. In this work we use sum-of-squares (SOS) methods to study the stability of an interconnected system by computing the vector Lyapunov functions as well as the comparison equations. While this approach is applicable to any generic dynamical system, we choose a randomly generated network of modified [1] Van der Pol oscillators for illustration. This network is decomposed into many interacting subsystems and each subsystem parameters are chosen so that individually each subsystem is stable, when the disturbances from neighbors are zero. SOS based expanding interior algorithm (see Jarvis-Wloszek (2003); Anghel et al. (2013)) is used to obtain estimate of region of attraction as sub-level sets of polynomial Lyapunov functions for each such subsystem. Finally SOS optimization is used to compute the stabilizing control policies, based on linear comparison systems, such that the closed-loop network is exponentially stable under initial disturbances.

Following some brief background in Section 2 we formulate the control design problem in Section 3. The sum-of-squares based distributed control algorithm is proposed in Section 4. In Section 5 we illustrate the control design on a network of Van der Pol systems, before concluding the article in Section 6.

2. PRELIMINARIES

2.1 Stability and Control of Nonlinear Systems

Let us consider the dynamical systems of the form

$$\dot{x}(t) = f(x(t)) + u_t, \quad t \geq 0, \; f(0) = 0, \tag{1}$$

where $x \in \mathbb{R}^n$ are the states, $u_t \in \mathbb{R}^n$ are the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and the origin is an equilibrium point [2] of the 'free' system, i.e. the system with no control ($u_t \equiv 0$). Let us first review the important concepts on stability of the equilibrium point of the 'free' system.

Definition 1. The equilibrium point at the origin is called asymptotically stable in a domain $\mathscr{D} \subseteq \mathbb{R}^n$, $0 \in \mathscr{D}$, if

$$\|x(0)\|_2 \in \mathscr{D} \implies \lim_{t\to\infty} \|x(t)\|_2 = 0,$$

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[1] We choose the Van der Pol 'oscillator' parameters in such a way that these have a stable equilibrium at origin.

[2] State variables can be shifted to move any equilibrium point to the origin.

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and it is exponentially stable if there exists $b,c>0$ such that

$$\|x(0)\|_2\in\mathscr{D}\Longrightarrow\|x(t)\|_2<ce^{-bt}\|x(0)\|_2\;\;\forall t\geq 0\,.$$

Lyapunov's first or direct method (see Lyapunov (1892); Slotine et al. (1991)) can give a sufficient condition of stability through the construction a certain positive definite function.

Theorem 1. If there exists a domain $\mathscr{D}\in\mathbb{R}^n$, $0\in\mathscr{D}$, and a continuously differentiable positive definite function $\tilde{V}:\mathscr{D}\rightarrow\mathbb{R}$, called the 'Lyapunov function' (LF), then the equilibrium point of the 'free' system at the origin is asymptotically stable if $\nabla\tilde{V}^Tf(x)$ is negative definite in $\mathscr{D}$, and is exponentially stable if $\nabla\tilde{V}^Tf(x)\leq -c\tilde{V}\;\forall x\in\mathscr{D}$, for some $c>0$.

When there exists such a $\tilde{V}(x)$, the region of attraction (ROA) of the equilibrium point at the origin can be estimated as

$$\mathscr{R}:=\{x\in\mathscr{D}\,|\,V(x)\leq 1\}\,, \tag{2a}$$

$$\text{where, } V(x)=\tilde{V}(x)/\gamma^{max}, \text{ and} \tag{2b}$$

$$\gamma^{max}:=\arg\max_{\gamma}\;\left\{x\in\mathbb{R}^n\,\middle|\,\tilde{V}(x)\leq\gamma\right\}\subseteq\mathscr{D}. \tag{2c}$$

For systems under some control action u_t, the notion of 'stabilizability' becomes important. Specifically, we are interested in state-feedback control of the form $u_t=u_t(x)$.

Definition 2. The system (1) is called (exponentially) stabilizable if there exists a control policy $u_t=u_t(x)$, $t\geq 0$, such that the origin of the closed-loop system is (exponentially) stable, in which case u_t is called a (exponentially) stabilizing control.

Courtesy to the works of Artstein (1983) and Sontag (1989), the concept of 'control Lyapunov functions' has been useful in the context of stabilizability.

Definition 3. A continuously differentiable positive definite function $V_c:\mathbb{R}^n\rightarrow\mathbb{R}$ is called a 'control Lyapunov function' (CLF) if for each $x\in\mathbb{R}^n\backslash\{0\}$, there exists a control u_t such that $\nabla V_c^T(f(x)+u_t)<0$.

Similar definition holds for 'exponentially stabilizing' CLFs (see Ames et al. (2014); Zhang et al. (2009)). CLFs can easily accommodate 'optimality' in the control policies as well (see Freeman and Kokotovic (2008)). However, as with the LFs, it is often very difficult to find a CLF for a given system.

2.2 Sum-of-Squares and Positivstellensatz Theorem

In recent years, sum-of-squares (SOS) based optimization techniques have been successfully used in constructing LFs by restricting the search space to sum-of-squares polynomials (see Jarvis-Wloszek (2003); Parrilo (2000); Tan (2006); Anghel et al. (2013)). Let us denote by $\mathbb{R}[x]$ the ring of all polynomials in $x\in\mathbb{R}^n$. Then,

Definition 4. A multivariate polynomial $p\in\mathbb{R}[x]$, $x\in\mathbb{R}^n$, is called a sum-of-squares (SOS) if there exists $h_i\in\mathbb{R}[x]$, $i\in\{1,\ldots,s\}$, for some finite s, such that $p(x)=\sum_{i=1}^s h_i^2(x)$. Further, the ring of all such SOS polynomials is denoted by $\Sigma[x]$.

Checking if $p\in\mathbb{R}[x]$ is an SOS is a semi-definite problem which can be solved with a MATLAB® toolbox SOSTOOLS (see Papachristodoulou et al. (2013); Papachristodoulou and Prajna (2005)) along with a semidefinite programming solver such as SeDuMi (see Sturm (1999)). SOS technique can be used to search for polynomial LFs, by translating the conditions in Theorem 1 to equivalent SOS conditions (see Jarvis-Wloszek (2003); Wloszek et al. (2005); Prajna et al. (2005)). An important result from algebraic geometry called Putinar's Positivstellensatz theorem [3] (see Putinar (1993); Lasserre (2009)) helps in translating the SOS conditions into SOS feasibility problems.

Theorem 2. Let $\mathscr{K}=\{x\in\mathbb{R}^n\,|\,k_1(x)\geq 0,\ldots,k_m(x)\geq 0\}$ be a compact set, where $k_j\in\mathbb{R}[x]$, $\forall j\in\{1,\ldots,m\}$. Suppose there exists a $\mu\in\left\{\sigma_0+\sum_{j=1}^m\sigma_jk_j\,\middle|\,\sigma_0,\sigma_j\in\Sigma[x],\forall j\right\}$ such that $\{x\in\mathbb{R}^n|\,\mu(x)\geq 0\}$ is compact. Then, if $p(x)>0\;\forall x\in\mathscr{K}$, then $p\in\left\{\sigma_0+\sum_j\sigma_jk_j\middle|\,\sigma_0,\sigma_j\in\Sigma[x],\forall j\right\}$.

In many cases, especially for the $k_i\forall i$ used throughout this work, a μ satisfying the conditions in Theorem 2 is guaranteed to exist (see Lasserre (2009)), and need not be searched for.

2.3 Linear Comparison Principle

Before finishing this section, let us take a look at a nice result on the ordinary differential equations which helps form the framework of stability analysis of inter-connected systems via vector LFs. Noting that all the elements of the vector e^{At}, $t\geq 0$, where $A=[a_{ij}]\in\mathbb{R}^{m\times m}$, are non-negative if and only if $a_{ij}\geq 0, i\neq j$, the authors in Beckenbach and Bellman (1961); Bellman (1962) proposed the following result:

Lemma 1. Let $A=[a_{ij}]\in\mathbb{R}^{m\times m}$ have only non-negative off-diagonal elements, i.e. $a_{ij}\geq 0,\; i\neq j$. Then

$$\dot{v}(t)\leq A\,v(t),\;t\geq 0,\;v\in\mathbb{R}^n,\;v(0)=v_0, \tag{3}$$

implies $v(t)\leq r(t)$, $\forall t\geq 0$, where

$$\dot{r}(t)=A\,r(t),\;t\geq 0,\;r\in\mathbb{R}^n,\;r(0)=v(0)=v_0. \tag{4}$$

This result will henceforth be referred to as the 'linear comparison principle' and the differential equation in (4) as the 'comparison equation'.

3. PROBLEM DESCRIPTION

The problem of interest for this work is to find state-feedback control $u_t=u_t(x)$ that exponentially stabilizes a large nonlinear system (1). One approach could be to find a suitable CLF (Definition 3), using computational methods, e.g. SOS technique. However, as noted in Anderson and Papachristodoulou (2012), such an approach will quickly become intractable as the system size increases. Instead, we seek distributed stabilizing control policies by modeling the large dynamical system as an inter-connected network of m (≥ 2) interacting subsystems,

$$\forall i=1,2,\ldots,m,$$

$$\mathscr{S}_i:\;\dot{x}_i=f_i(x_i)+u_{t,i}+g_i(x),\;x_i\in\mathbb{R}^{n_i},\;x\in\mathbb{R}^n \tag{5a}$$

$$f_i(0)=0, \tag{5b}$$

$$g_i(\hat{x}_i)=0,\;\forall\hat{x}_i\in\left\{x\in\mathbb{R}^n\,\middle|\,x_j=0,\forall j\neq i\right\} \tag{5c}$$

$$\text{where, } x=\bigcup_{j=1}^m\{x_j\}\,,\text{ and } n\leq\sum_{j=1}^m n_j\,. \tag{5d}$$

We assume that the isolated 'free' subsystem dynamics $f_i\in\mathbb{R}[x_i]^{n_i}$, and the neighbor interactions $g_i\in\mathbb{R}[x]^{n_i}$ are vectors of polynomials. Further, $u_{t,i}=u_{t,i}(x_i)$ is a time-dependent local state-feedback control policy, with each $u_{t,i}\in\mathbb{R}[x_i]^{n_i}\;\forall t$. It is assumed that the 'free' isolated subsystems as well as the 'free' full system are (locally) exponentially stable. Note that, we allow over-lapping decomposition in which subsystems can

[3] Refer to Lasserre (2009) for other versions of the Positivstellensatz theorem.