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IFAC-PapersOnLine 48-13 (2015) 252–257

Robustness and Algebraic Connectivity of Random Interdependent Networks ⋆

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Abstract: We investigate certain structural properties of random interdependent networks. We start by studying a property known as r-robustness, which is a strong indicator of the ability of a network to tolerate structural perturbations and dynamical attacks. We show that random k-partite graphs exhibit a threshold for r-robustness, and that this threshold is the same as the one for the graph to have minimum degree r. We then extend this characterization to random interdependent networks with arbitrary intra-layer topologies. Finally, we characterize the algebraic connectivity of such networks, and provide an asymptotically tight rate of growth of this quantity for a certain range of inter-layer edge formation probabilities. Our results arise from a characterization of the isoperimetric constant of random interdependent networks, and yield new insights into the structure and robustness properties of such networks.

Keywords: Random Interdependent Networks, Robustness, Algebraic Connectivity, Isoperimetric Constant.

1. INTRODUCTION

There is an increasing realization that many large-scale networks are really "networks-of-networks," consisting of interdependencies between different subnetworks (e.g., coupled cyber and physical networks) [Shahrivar and Sundaram (2013); Radicchi and Arenas (2013); Martin-Hernandez et al. (2014); Schneider et al. (2013)]. Due to the prevalence of such networks, their robustness to intentional disruption or natural malfunctions has started to attract attention by a variety of researchers [Schneider et al. (2013); Yagan et al. (2012); Parandehgheibi and Modiano (2013)]. In this paper, we contribute to the understanding of interdependent networks by studying the graph-theoretic properties of *r-robustness* and *algebraic connectivity* in such networks. As we will describe further in the next section, r-robustness has strong connotations for the ability of networks to withstand structural and dynamical disruptions: it guarantees that the network will remain connected even if up to $r-1$ nodes are removed from the neighborhood of *every* node in the network, and facilitates certain consensus dynamics that are resilient to adversarial nodes [LeBlanc et al. (2013); Zhao et al. (2014); Dibaji and Ishii (2015); Zhang et al. (2015); Vaidya et al. (2012)]. The algebraic connectivity of a network is the second smallest eigenvalue of the Laplacian of that network, and plays a key role in the speed of certain diffusion dynamics [Olfati-Saber et al. (2007)].

We focus our analysis on a class of *random interdependent networks* consisting of k *layers* (or subnetworks), where each edge between nodes in different layers is present independently with certain probability p. Our model is fairly general in that we make no assumption on the topologies within the layers, and captures Erdos-Renyi graphs and random k-partite graphs as special cases. We identify a bound p_r for the probability of inter-layer edge formation p such that for $p > p_r$, random interdependent networks with arbitrary intra-layer topologies are guaranteed to be r-robust asymptotically almost surely. For the special case of k-partite random graphs, we show that this p_r is tight (i.e., it forms a threshold for the property of r-robustness), and furthermore, is *also* the threshold for the minimum degree of the network to be r. This is a potentially surprising result, given that r-robustness is a significantly stronger graph property than r-minimum-degree. Recent work has shown that these properties also share thresholds in Erdos-Renyi random graphs [Zhang et al. (2015)] and random intersection graphs [Zhao et al. (2014)], and our work in this paper adds random k-partite graphs to this list.

Next, we show that when the inter-layer edge formation probability p satisfies a certain condition, both the robustness parameter and the algebraic connectivity of the network grow as $\Theta(np)$ asymptotically almost surely (where n is the number of nodes in each layer), *regardless* of the topologies within the layers. Given the key role of algebraic connectivity in the speed of consensus dynamics on networks [Olfati-Saber et al. (2007)], our analysis demonstrates the importance of the edges that

⋆ This material is based upon work supported in part by the Natural Sciences and Engineering Research Council of Canada.

10.1016/j.ifacol.2015.10.339

connect different communities in the network in terms of facilitating information spreading, in line with classical findings in the sociology literature [Granovetter (1973)]. Our result on algebraic connectivity of random interdependent networks is also applicable to the stochastic block model or planted partition model that has been widely studied in the machine learning literature [Feige and Kilian (2001); Abbe et al. (2014); Dasgupta et al. (2013); McSherry (2001)]. While we consider arbitrary intra-layer topologies, in the planted partition model it is assumed that the intra-layer edges are also placed randomly with a certain probability. Furthermore, the lower bound that we obtain here for $\lambda_2(L)$ is tighter than the lower bounds obtained in the existing planted partition literature for the range of edge formation probabilities that we consider [Feige and Kilian (2001); Dasgupta et al. (2013)]. Both our robustness and algebraic connectivity bounds arise from a characterization that we provide of the isoperimetric constant of random interdependent networks.

2. GRAPH DEFINITIONS AND BACKGROUND

An undirected graph is denoted by $G = (V, E)$ where V is the set of vertices (or nodes) and $E \subseteq V \times V$ is the set of edges. We denote the set $\mathcal{N}_i = \{v_j \in V \mid (v_i, v_j) \in E\}$ as the *neighbors* of node $v_i \in V$ in graph G. The *degree* of node v_i is $d_i = |\mathcal{N}_i|$, and d_{min} and d_{max} are the minimum and maximum degrees of the nodes in the graph, respectively. A graph $G' = (V', E')$ is called a subgraph of $G = (V, E)$, denoted as $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E \cap \{V' \times V'\}$. For an integer $k \in \mathbb{Z}_{\geq 2}$, a graph G is k-partite if its vertex set can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that there are no edges between nodes within any of those sets.

The *edge-boundary* of a set of nodes $S \subset V$ is given by $\partial S = \{(v_i, v_j) \in E \mid v_i \in S, v_j \in V \setminus S\}$. The *isoperimetric constant* of G is defined as [Chung (1997)]

$$i(G) \triangleq \min_{A \subset V, |A| \leq \frac{n}{2}} \frac{|\partial A|}{|A|}. \quad (1)$$

By choosing A as the vertex with the smallest degree we obtain $i(G) \leq d_{min}$.

The *adjacency matrix* for the graph is a matrix $A \in \{0, 1\}^{n \times n}$ whose (i, j) entry is 1 if $(v_i, v_j) \in \mathcal{E}$, and zero otherwise. The *Laplacian matrix* for the graph is given by $L = D - A$, where D is the degree matrix with $D = \text{diag}(d_1, d_2, \ldots, d_n)$. For an undirected graph G, the Laplacian L is a symmetric matrix with real eigenvalues that can be ordered sequentially as $0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_n(L) \leq 2d_{max}$. The second smallest eigenvalue $\lambda_2(L)$ is termed the *algebraic connectivity* of the graph and satisfies the bound [Chung (1997)]

$$\frac{i(G)^2}{2d_{max}} \leq \lambda_2(L) \leq 2i(G). \quad (2)$$

2.1 r-Robustness of Networks

Early work on the robustness of networks to structural and dynamical disruptions focused on the notion of *graph-connectivity*, defined as the smallest number of nodes that have to be removed to disconnect the network [West (2001)]. A network is said to be r-connected if it has connectivity at least r. In this paper, we will consider a stronger metric known as r-robustness, given by the following definition.

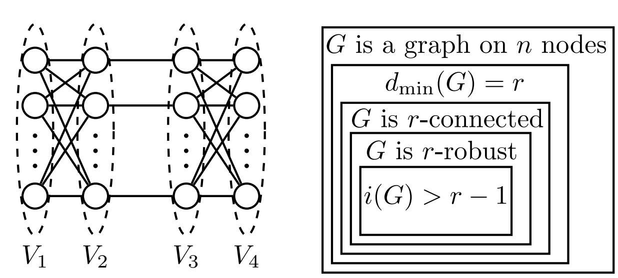
Definition 1. (LeBlanc et al. (2013)). Let $r \in \mathbb{N}$. A subset S of nodes in the graph $G = (V, E)$ is said to be *r-reachable* if there exists a node $v_i \in S$ such that $|\mathcal{N}_i \setminus S| \geq r$. A graph $G = (V, E)$ is said to be *r-robust* if for every pair of nonempty, disjoint subsets of V, at least one of them is r-reachable.

Simply put, an r-reachable set contains a node that has r neighbors outside that set, and an r-robust graph has the property that no matter how one chooses two disjoint nonempty sets, at least one of those sets is r-reachable. This notion carries the following important implications:

- If network G is r-robust, then it is at least r-connected and has minimum degree at least r [LeBlanc et al. (2013)].
- An r-robust network remains connected even after removing up to $r - 1$ nodes from the neighborhood of *every* remaining node [Zhang et al. (2015)].
- Consider the following class of consensus dynamics where each node starts with a scalar real value. At each iteration, it discards the highest F and lowest F values in its neighborhood (for some $F \in \mathbb{N}$), and updates its value as a weighted average of the remaining values. It was shown in [LeBlanc et al. (2013); Dibaji and Ishii (2015)] that if the network is $(2F + 1)$-robust, all nodes that follow these dynamics will reach consensus even if there are up to F arbitrarily behaving malicious nodes in the neighborhood of *every* normal node.

Thus, r-robustness is a stronger property than r-minimum-degree and r-connectivity. Indeed, the gap between the robustness and connectivity (and minimum degree) parameters can be arbitrarily large, as illustrated by the bipartite graph shown in Fig. 1a. That graph has minimum degree $n/4$ and connectivity $n/4$. However, if we consider the disjoint subsets $V_1 \cup V_2$ and $V_3 \cup V_4$, neither one of those sets contains a node that has more than 1 neighbor outside its own set. Thus, this graph is only 1-robust.

The following result shows that the isoperimetric constant $i(G)$ defined in (1) provides a lower bound on the robustness parameter.



(a) (b)

Fig. 1. (a) Graph $G = (V, E)$ with $V = V_1 \cup V_2 \cup V_3 \cup V_4$ and $|V_i| = \frac{n}{4}$, $1 \leq i \leq 4$. All of the nodes in V_1 (V_3) are connected to all of the nodes in V_2 (V_4). Furthermore, there is a one to one connection between nodes in V_2 and nodes in V_3. (b) Relationships between different notions of robustness.