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Distributed Consensus with Multiple Equilibria in Continuous-Time Multi-Agent Networks under Undirected Topologies ⋆

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Abstract: As the networks get more complex nowadays with many interconnected components, it is necessary to develop distributed scalable algorithms so as to minimize the computation required in decision making in such large scale systems. In this context, this paper addresses the problem of multiple equilibrium consensus of multi-agent systems in undirected networks having fixed or switching topologies. The convergence analysis is based upon algebraic graph theory and switched system theory. Finally, some numerical examples are given to illustrate the theoretical results.

Keywords: distributed consensus, matrix theory, multiple equilibria, switching networks, optimization.

1. INTRODUCTION

In the recent years, there has been a vast amount of research on the coordination control of multi agent networks. Distributed coordination control problems in networks of dynamic agents are addressed in various forms such as consensus, optimization, task assignment and formation control. The consensus problem, where a group of agents agree upon certain quantities of interest while exchanging information among agents, is one of the most important in this context. (Beard et. al. (2002); Fax and Murray (2004); Cihan and Akar (2015); Tsitsiklis et. al. (1986); Blondel et al. (2005); Vicsek et. al. (1995); Jadbabaie et. al. (2003); Moreau (2005); Ren and Beard (2005); Olfati-Saber et al. (2007); Akar and Shorten (2008); Cao et. al. (2008); Gazi (2008)).

Olfati-Saber and Murray (2004) propose a continuous-time consensus algorithm and investigate its convergence properties for fixed and switching topologies with time delays. Convergence of the algorithm is established by using Lyapunov theory given that the underlying graph topology is balanced and strongly connected.

In 2003, an analysis for the convergence properties of a multi-agent system modeled with undirected graphs is given by Jadbabaie et. al. (2003). They show that in order to achieve convergence, there must be an infinite sequence of contiguous nonempty bounded time intervals across which all agents are linked together. The proof of the consensus algorithm depends on the convergence of stochastic matrix products known as Wolfowitz theorem. Subsequently, this work is extended to directed graphs in Moreau (2005); Ren and Beard (2005) and it is shown that consensus is achieved if and only if the underlying graph has a directed spanning tree.

The aforementioned studies deal with multi-agent systems that reach single a equilibrium state, known also as traditional consensus. However, in distributed control of multi-agent systems, agreement value can differ according to the tasks, working space or even time itself.

Consensus on multiple equilibrium states has many potential research areas especially when exploring social networks, collective behavior of organisms in the nature or robotics. For example, in an election, the participants give votes which vary according to the relationship they have with people around them. Therefore, the network consists of different opinions raising more than one equilibrium state (Li et. al. (2013)). An animal group can also be given as a network example with multiple consensus equilibrium states in which the ants select new nests according to some metrics. Therefore, it is required to design and analyze distributed scalable consensus protocols for a multi-agent system so as to operate in faulty and uncertain environments. Although, single point consensus problems have been well addressed in the literature, clustering/group consensus algorithms are only recently receiving attention (Yu and Wang (2012); Tan et. al. (2011); Xia and Cao (2011); Qin and Yu (2013); Han et. al. (2013b); Yu and Qin (2014); Chen et. al. (2011); Han et. al. (2013)).

All of the above work impose strong restrictions on graph topology and/or coupling coefficients which are not applicable to real–life problems. For instance, they have a common assumption that the interaction between any two subgroups (clusters) has to be balanced. Moreover, the papers deal with a network which is artificially divided into multiple subnetworks.

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Unlike the aforementioned studies in which the clusters are determined a priori, we focus on the multiple equilibrium consensus problem for any given network, which is an extension of the classical consensus problem. The problem is handled in the continuous-time setting. We show that under certain assumptions the network achieves multiple equilibrium consensus states under switching topologies, by extending the results in (Jadbabaie et. al. (2003); Ren and Beard (2005)).

The organization of this paper is as follows. In Section 2, we introduce the continuous-time consensus protocol, and formulate the problem of multi equilibrium consensus. In Section 3, we analyze its convergence properties using tools of graph theory, matrix theory and switched systems. In Section 4, the theoretical results are illustrated with examples, while Section 5 concludes the paper.

2. PROBLEM FORMULATION & MATHEMATICAL PRELIMINARIES

In this section, we present mathematical preliminaries and definitions that are used to study distributed consensus. Additionally, the problem of distributed consensus with multiple equilibrium states is formulated for networks with fixed and time-varying topologies.

2.1 Graph Theoretic Concepts

Let $G = (V, E)$ be the graph that represents a network consisting of n agents. The elements of $V = \{v_1, v_2, \ldots, v_n\}$ are defined as the nodes, and elements of $E \subseteq V \times V$ are the edges of the given graph. The node indices take values in a finite index set $\mathcal{I} = \{1, 2, \ldots, n\}$. An edge which shows information exchange among agents in G is denoted by $e_{ij} = (v_j, v_i)$ and represented as an arrow from node i to j.

Two nodes i, j of G are *neighbors*, if e_{ij} is an edge of G. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in V : (v_j, v_i) \in E\}$. The considered graph is said to be undirected, if for all $i, j \in V : (v_i, v_j) \in E$ implies $(v_j, v_i) \in E$. Otherwise, the graph is directed.

A *directed path* in a digraph is defined as the sequence of edges $(e_1, e_2), (e_2, e_3), \ldots$ which connects a sequence of nodes. An undirected graph is said to be *connected* if there is a path from every node to every other node. Graph G is said to be *acyclic* if it does not contain a cycle. A *tree* is defined as a connected acyclic graph. A *spanning tree* of a graph is defined as a tree containing all the vertices of G. The node of degree 1 except the *root* in a tree is called as its leaf. The parent node in a rooted tree is the node connected to it on the path to the root. Every node except the root has at most one parent.

For the case that the topology is time-varying, the network can be described by a dynamic graph $\mathbb{G} = (V, E(t))$. The network topology will switch among a set of topologies given by $\mathbb{G} = \{\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_N\}$. The union of graphs $\{\mathbb{G}_1, \ldots, \mathbb{G}_m\} \subset \mathbb{G}$ is referred to as an undirected graph with nodes given by v_i, $i \in \mathcal{I}$ and edge set defined by the union of edge sets $\mathbb{G}_j$, $j = 1, \ldots, m$. The notions agent and node will be used interchangeably throughout the paper.

2.2 Distributed Consensus Protocol

Let $G = (V, E)$ be the graph and $x_i(t) \in R$ be the state value of the i-th agent in the network. The state value (that could be clocks, voltage, or heading angle in a physical system) of each agent is updated according to the following continuous-time distributed consensus protocol

$$\dot{x}_i(t) = \sum_{v_j \in N_i} a_{ij}(t)(x_j(t) - x_i(t)) \tag{1}$$

where $a_{ij}(t)$ denotes the (i, j) th entry of the corresponding weighted adjancency matrix at time t, that satisfies the following assumption.

Assumption 1. If there is no link (edge) between the agents i and j, then $a_{ij}(t) = 0$. Otherwise, for some positive parameter δ, $a_{ij}(t) \geq \delta$.

Protocol (1) can be expressed as

$$\dot{x}(t) = -L(t)x(t), x(0) = x_0 \tag{2}$$

where $L(t)$ denotes graph Laplacian at each time t and $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\top$. The graph Laplacian becomes time-invariant for fixed networks. The elements of the Laplacian matrix $L = [l_{ij}]$ are given by

$$l_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij}, & \text{if } i = j \\ -a_{ij}, & \text{if } i \neq j \end{cases} \tag{3}$$

By definition, each row of Laplacian matrix adds up to zero. Therefore, L has a zero eigenvalue associated with the right eigenvector $\mathbf{1} = [1 \ldots 1]^\top \in R^n$. In many studies, it is often assumed that network topology is time-invariant under ideal communication channels. One can examine the stability properties of the protocol (2) for a fixed topology by considering the location of the eigenvalues of the graph Laplacian. However, network topology may change dynamically due to the creation of new links and/or breakage of existing links.

The following results summarizes the convergence analysis of the protocol (1) for fixed topology networks.

Lemma 1. (Ren and Beard (2005), Lemma 3.3). Given a matrix $L = [l_{ij}]$, where $l_{ii} \geq 0$, $l_{ij} \leq 0$, $\forall i \neq j$, and $\sum_{j=1}^{n} l_{ij} = 0$, $i, j \in \mathcal{I}$ for each j, L has at least one zero eigenvalue and the rest of the nonzero eigenvalues are positive and real. Additionally, L has a simple zero eigenvalue if and only if the underlying digraph has a spanning tree.

Remark 1. If the undirected graph G is connected, then its Laplacian matrix has a simple zero eigenvalue and all its other eigenvalues are positive and real.

Lemma 2. (Olfati-Saber and Murray (2004), Th. 4). Protocol (1) achieves consensus asymptotically for all initial conditions if and only if the associated undirected graph G is connected.

Lemma 3. (Ren and Beard (2005), Th. 3.8). For fixed network topology with constant weighting coefficients, the continuous-time protocol (1) achieves consensus if and only if the related graph has a spanning tree.