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Flatness of Mechanical Systems with 3 Degrees of Freedom

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Abstract: We study flatness of two-input mechanical systems with 3 degrees of freedom. Our goal is to describe the geometry of this class of systems from the point of view of differential weight. We completely characterize flatness of differential weight 8 and 9. We show that a mechanical systems with 3 degrees of freedom is flat of differential weight 8 or 9 if and only if it is config-flat or vel-flat of differential weight 8 or 9 and we describe config-flatness of differential weight 10.

Keywords: mechanical systems, flatness, differential weight, normal forms.

1. INTRODUCTION

In this paper, we study flatness of mechanical control systems of the form

$$(\mathcal{MS}) \begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = -\Gamma_i^{jk}(x) y_j y_k + d_i^j(x) y_j + e_i(x) + \sum_{r=1}^{m} u_r g_{ri}(x) \end{cases}$$

with $1 \leq i \leq n$ and where $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ are local coordinates on the tangent bundle TQ of the configuration manifold Q and $u = (u_1, \ldots, u_m)$ are inputs of the system. The summation convention is used, except for terms involving controls. The expression $\Gamma_i^{jk}(x)y_j y_k$ corresponds to Coriolis and centrifugal terms. The terms $d_i^j(x)y_j$ correspond to forces linear with respect to velocities, like dissipative forces, $e(x)$ represents an uncontrolled force and $g_1, \ldots, g_m$ controlled forces acting on the system. Mechanical control systems form an important class of control systems that has attracted a lot of attention because of its many applications in real life. They form a natural bridge between mechanics and control theory and are studied, for instance, in Ortega et al. (1998); Bloch (2003); Bullo and Lewis (2004); Ricardo and Respondek (2010).

Two important problems in control theory and, in particular, in the study of mechanical control systems, are trajectory tracking and constructive controllability or trajectory generation. A class of systems for which these problems are particularly easy to solve are so-called flat systems. The notion of flatness has been introduced in control theory in the 1990's by Fliess, Lévine, Martin and Rouchon in Fliess et al. (1995, 1992) (see also Isidori et al. (1986); Jakubczyk (1993); Martin (1992); Pomet (1995) and Fliess et al. (1999); Pomet (1997); Pereira da Silva (2001); Martin et al. (2003); Schlacher and Schoeberl (2007); Lévine (2009)). The fundamental property of flat systems is that all their solutions may be parametrized by m functions and their time-derivatives, m being the number of controls. More precisely, the control system $\Xi : \dot{\xi} = F(\xi, u)$, $\xi \in X \subset \mathbb{R}^N$, $u \in U \subset \mathbb{R}^m$, is *flat* if we can find m functions, $\varphi_i(\xi, u, \ldots, u^{(p)})$, for some $p \geq 0$, called *flat outputs*, such that

$$\xi = \gamma(\varphi, \ldots, \varphi^{(s)}) \text{ and } u = \delta(\varphi, \ldots, \varphi^{(s)}), \qquad (1)$$

for a certain integer s, where $\varphi = (\varphi_1, \ldots, \varphi_m)$. Therefore, all state and control variables can be determined from the flat outputs without integration and all trajectories of the system can be completely parametrized.

Although flatness has been intensively studied, the problem of giving necessary and sufficient verifiable conditions in order to determine if a control systems is flat is largely open. A number of special cases, however, are well understood. For instance, it is well known that systems linearizable via invertible static feedback are flat and their description (1) uses the minimal possible, which is $N + m$, number of time-derivatives of the components of flat outputs φ_i, where N is the state dimension and m is the number of controls. In fact, flat systems can be seen as a generalization of linear systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see Fliess et al. (1995, 1992); Martin (1992); Pomet (1997). For any flat system, that is not static feedback linearizable, the minimal number of derivatives needed to express ξ and u (that we will call the differential weight) is thus bigger than $N + m$ and measures actually the smallest possible dimension of a precompensator linearizing dynamically the system.

We gave a geometric characterization of flat systems of differential weight $N + 3$, for systems with 2 controls, and of differential weight $N + m + 1$, for systems with $m \geq 3$ controls in, respectively, Nicolau and Respondek (2013a) and Nicolau and Respondek (2013b) (see also Nicolau and Respondek (2014a,b)). Those systems form a particular class of flat systems: they become static feedback linearizable after a one-fold prolongation of a suitably chosen control.

Many interesting examples of mechanical system are flat and for many of them, flat outputs depend on the configuration variables x only but not on their derivatives (velocities) y. Such flat mechanical system, i.e., for which all functions φ_i, for $1 \leq i \leq m$, depend on the configurations x only, are called config-flat, see Murray et al. (1995). Rathinam and Murray (1998) proposed a characterization

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of config-flatness of Lagrangian systems underactuated by one control (with n degrees of freedom and $n-1$ controls) in terms of the Riemannian metric corresponding to kinetic energy which essentially determines the config-flatness of the considered class of systems. Sato and Iwai (2012) extended that result by giving a necessary condition for Lagrangian control systems to be config-flat and two types of sufficient conditions for Lagrangian control systems underactuated by 2 controls to be config-flat. Knoll and Robenack (2014) analyzed the config-flatness of the linear approximation of Lagrangian mechanical systems and showed that for systems where friction is either absent or can be compensated by the inputs, flatness implies config-flatness. Some of the mechanical systems that we are studying here fall into the class considered in Rathinam and Murray (1998), but the goal of this paper is to understand the geometry of mechanical control systems with 3 degrees of freedom from the point of view of differential weight, to decide when a mechanical system of that class is config-flat and when it is not config-flat, but vel-flat (i.e., where the components φ_i of the flat output φ involve not only the configurations x, but also the velocities y), how the vel-flat outputs depend on the velocities and how flatness is related to static and dynamic feedback linearization and to the mechanical structure of the system (for instance, we will be interested in the compatibility of the transformations linearizing the system and the mechanical structure).

The considered systems have 6 states and 2 controls so the minimal possible weight is 8, which we study first. Then we describe flatness of differential weight 9. We show that a system $(\mathcal{MS})$ with 3 degrees of freedom is flat of differential weight 8 (resp. of differential weight 9) if and only if it is config-flat or vel-flat of differential weight 8 (resp. of differential weight 9). We give feedback invariant conditions describing general flatness of differential weight 8 (resp. of differential weight 9) and then we complete them by some geometric conditions in order to obtain a characterization of config-flatness. Finally, we study config-flatness of differential weight 10. For all cases we present normal forms and our results cover completely the config-flatness of the considered class of systems. Indeed, we state that a mechanical system with 3 degrees of freedom is config-flat if and only if it is config-flat of differential weight lower or equal to 10. Therefore two-input config-flat mechanical system with 6 states can be static feedback linearizable or dynamic feedback linearizable via the application of a precompensator of dimension at most two. We show that the dynamic precompensator corresponds to a one- or two-fold prolongation of a suitably chosen control (which is the simplest dynamic feedback) and moreover this control is compatible with the mechanical structure of the system (see Section 2.2 for the definition of feedback transformations compatible with the mechanical structure).

The paper is organized as follows. In Section 2, we provide some preliminary notions, in particular we recall the definition of flatness and that of differential weight of a flat system and discuss the equivalence (under a diffeomorphism of configuration manifolds and an invertible feedback transformation) of two mechanical control systems. In Section 3, we give our main results. We characterize flat mechanical system with 3 degrees of freedom. We illustrate our results by two examples in Section 4.

2. PRELIMINARIES

2.1 Flatness

Consider the nonlinear control system $\Xi : \dot{\xi} = F(\xi, u)$, where ξ is the state defined on a open subset X of $\mathbb{R}^N$ and u is the control taking values in an open subset U of $\mathbb{R}^m$ (more generally, an N-dimensional manifold X and an m-dimensional manifold U, resp.). The dynamics F are smooth and the word smooth will always mean $\mathcal{C}^\infty$-smooth. Fix an integer $l \geq -1$ and denote $U^l = U \times \mathbb{R}^{ml}$ and $\bar{u}^l = (u, \dot{u}, \ldots, u^{(l)})$. For $l = -1$, the set U^{-1} is empty and $\bar{u}^{-1}$ in an empty sequence.

Definition 2.1. The system $\Xi : \dot{\xi} = F(\xi, u)$ is *flat* at $(\xi_0, \bar{u}_0^l) \in X \times U^l$, for $l \geq -1$, if there exists a neighborhood $\mathcal{O}^l$ of $(\xi_0, \bar{u}_0^l)$ and m smooth functions $\varphi_i = \varphi_i(\xi, u, \dot{u}, \ldots, u^{(l)})$, $1 \leq i \leq m$, defined in $\mathcal{O}^l$, having the following property: there exist an integer s and smooth functions γ_i, $1 \leq i \leq N$, and δ_j, $1 \leq j \leq m$, such that

$$\xi_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

along any trajectory $\xi(t)$ given by a control $u(t)$ that satisfy $(\xi(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called a *flat output*.

In the case of a flat mechanical system of the form $(\mathcal{MS})$, we have $\xi = (x, y)$, $N = 2n$ and if all functions φ_i, for $1 \leq i \leq m$, depend on the configuration variables x only, then we will say that the system is *config-flat*. If the components φ_i of the flat output φ involve not only the configurations x, but also the velocities y, then the system will be called *vel-flat*.

The minimal number of derivatives of components of a flat output, needed to express ξ and u, will be called the differential weight of that flat output and is formalized as follows. By definition, for any flat output φ of a flat system Ξ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{aligned} \xi &= \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}). \end{aligned}$$

Moreover, we can choose $(s_1, \ldots, s_m)$ such that if for any other m-tuple $(\tilde{s}_1, \ldots, \tilde{s}_m)$ we have

$$\begin{aligned} \xi &= \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}) \\ u &= \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}), \end{aligned}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$ (see Respondek (2003)). We will call $\sum_{i=1}^m (s_i + 1) = m + \sum_{i=1}^m s_i$ the differential weight of φ. A flat output of Ξ is called *minimal* if its differential weight is the lowest among all flat outputs of Ξ. We define the *differential weight* of a flat system to be equal to the differential weight of any of its minimal flat outputs.

2.2 Mechanical feedback equivalence

Let $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$ be two mechanical control systems. We say that they are mechanical feedback equivalent (*MF-equivalent*) (resp., *locally MF-equivalent* at points (x_0, y_0) and $(\tilde{x}_0, \tilde{y}_0)$) if they are static feedback equivalent (resp., locally static feedback-equivalent around points (x_0, y_0) and $(\tilde{x}_0, \tilde{y}_0)$) under an extended point transformation $\Phi = (\phi_1, \phi_2)$ of the form $\tilde{x} = \phi_1(x)$ and $\tilde{y} = \phi_2(x, y) = \mathrm{D}\phi_1(x)y$, and an invertible static feedback transformation $\tilde{u} = \tilde{\alpha}(x, y) + \tilde{\beta}(x)u$, with $\tilde{\beta}$ an invertible matrix and $\tilde{\alpha}$ quadratic with respect to y, where (x, y) and $(\tilde{x}, \tilde{y})$ (resp., u and $\tilde{u}$) are local coordinates (resp. controls) of $(\mathcal{MS})$ and $(\tilde{\mathcal{MS}})$. We will say that the system $(\mathcal{MS})$ is *MF-transformed* into $(\widetilde{\mathcal{MS}})$. Notice that in order to