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Widening the effect of Lie bracket motion: a semi-global approximation and control for nonholonomic systems using non-power series expansion ⋆

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Abstract: A nonlinear control system often has complicated input-to-state relationship, mainly due to its controllability structure in nonlinear sense. A difficult but challenging nature of such systems is that they can only be controlled using appropriate combination of *periodic* inputs, corresponding to the Lie brackets. However, the effect of so-called Lie bracket motion tends to be limited due to small choice of the input amplitudes. In this paper, focusing on a class of nonholonomic systems as typical examples, we propose to approximate the state displacement under periodic signals with larger amplitudes, and utilize the result to design periodic control input. The key of our approach is the use of suitable special function, such as the Bessel functions, for series expansion to predict the state displacement, as well as considering symmetry of the state space.

Keywords: nonlinear control, nonholonomic systems, holonomy, periodic control, averaging technique, mobile robot

1. INTRODUCTION

Nonlinear dynamical systems have been providing a lot of challenging topics to control engineers. A major peculiarity of nonlinear systems would be their complicated structure of controllability. In the case of linear time-invariant systems, the principle of superposition holds, the control inputs just go through linear combination of integrator chains, and their controllability is judged by rank condition of the controllability matrix. In contrast, nonlinear systems often have complicated input-to-state or input-to-output relationship; it is not easy to describe, in general, where the state variable would reach for the given control inputs. Controllability test for nonlinear systems are given in various forms of Lie algebra rank condition (Hermann and Krener (1977); Sussmann (1983); Aeyels (1983) depending on what we expect for controllability.

Nonholonomic systems form a class of such nonlinear systems, originally defined in classical mechanics as non-integrable mechanical constraints (Goldstein et al. (2002); Bloch (2003)). Most of nonholonomic systems have multiple control inputs that are coupled to each other, and some of them are controllable even without so-called drift term (say $f(\xi)$), thanks to the Lie bracket between the input vector-fields (such as $[g_1, g_2](\xi)$). It is also known that the system without drift term (driftless systems) are not stabilizable by any smooth time-invariant nonlinear feedback law Brockett (1983) even if they are controllable; this fact motivate us to pursue non-smooth approach (switched control, discontinuous control) or partially feedforward approach (time-varying control, periodic control).

A well-known basic principle for periodic control of nonholonomic systems is so-called *area rule*, which relates the Lie brackets with periodic control signals in the sense that the state change after one cycle of periodic control is approximately parallel to the corresponding Lie bracket and is proportional to the area enclosed by the periodic signal (Murray and Sastry (1993); Leonard and Krishnaprasad (1995)). However, this sort of approxomation is essentially based on Taylor series expansion, thus it does work well for periodic control with small amplitudes, or in the case of systems with drift term. From practical points of view, we often prefer to utilize full range of control amplitude as well as effect of the drift term to save control effort. In this paper, focusing on a typical example of nonholonomic system with drift term, we simply derive the system solution under periodic control using series expansion with Bessel functions, whose approximation is valid for the case of large amplitude.

This paper is organized as follows. In Section 2, we explain a motivating problem derived from unilateral rolling disk control. Section 3 describes a limitation of Lie bracket-based analysis. In Section 4, we give an explicit solution of the target system and a feedforward point-to-point control under given periodic control with several numerical examples. Concluding remarks are given in Section 5.

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2. MOTIVATION FROM CYLINDRICAL MOBILE ROBOT

This paper is motivated by the authors' previous work on cylindrical mobile robot (Hirano et al. (2013)). Fig. 1 shows a rolling cylinder put on the horizontal floor, where a point on the rim of the circular section is in contact with the floor (this motion is called *edge rolling*). R is the constant radius of the cylinder, (X_C, Y_C) indicates the position of the contact point, ψ is the heading angle and ϕ is the rotation(pitch) angle of the cylinder. The robot itself is supposed to be driven by actuating an internal pendulum. We assume that there is no slip nor slide between the cylinder and the floor, although its spin about the vertical axis passing through the contact point is freely allowed. If we assign the speed of the contact point $R\dot{\psi}$ as the control input u_1, and the heading angular velocity $\dot{\psi}$ as the other control input u_2, then the system can be viewed as a standard two-wheeled mobile robot (Fig.2). In the edge rolling, the contact point almost always moves along circular loci because of the gyroscopic effect generated by the gravity.

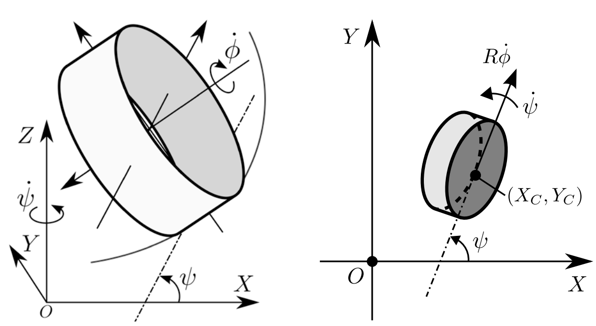

Fig. 1. Cylindrical rolling robot in edge-rolling motion

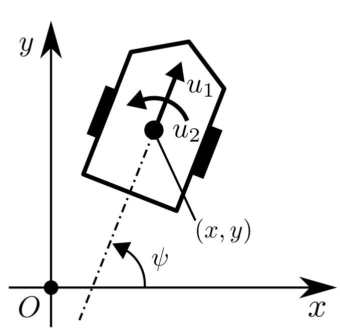

Fig. 2. Model of the simple two-wheeled robot

Note that, in order to maintain the edge rolling motion, the rotation speed $\dot{\phi}$ and the heading angular velocity should be *certain non-zero values*, say $\bar{u}_1, \bar{u}_2$, respectively. This motivates us to consider the following model of nonlinear state equation with drift, where the control inputs $u_1(t)$, $u_2(t)$ can be applied as deviation to the constant bias $\bar{u}_1, \bar{u}_2$.

$$\dot{\xi} = f(\xi) + g_1(\xi)u_1(t) + g_2(\xi)u_2(t), \tag{1}$$

$$\xi(t) = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}, f(\xi) = \begin{bmatrix} \bar{u}_1 \cos\psi \\ \bar{u}_1 \sin\psi \\ \bar{u}_2 \end{bmatrix},$$

$$g_1(\xi) = \begin{bmatrix} \cos\psi \\ \sin\psi \\ 0 \end{bmatrix}, g_2(\xi) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

3. EFFECT AND LIMITATION OF TAYLOR SERIES APPROACH

In the case of driftless nonlinear state equation

$$\dot{\xi} = g_1(\xi)u_1(t) + g_2(\xi)u_2(t), \tag{2}$$

an elementary approach to control is to apply a pair of periodic control signals as mentioned in the introduction. If we apply the following square periodic wave

$$(u_1(t), u_2(t)) = \begin{cases} (1, 0), & t \in [0, T) \\ (0, 1), & t \in [T, 2T) \\ (-1, 0), & t \in [2T, 3T) \\ (0, -1), & t \in [3T, 4T] \end{cases} \tag{3}$$

to the system(2), then the resulting displacement in the state space is given by

$$\xi(4T) = \xi(0) + T^2[g_1, g_2](\xi(0)) + O(T^3), \tag{4}$$

where the Lie bracket is defined as the vector-field

$$[g_1, g_2](\xi) := \frac{\partial g_2}{\partial \xi} g_1(\xi) - \frac{\partial g_1}{\partial \xi} g_2(\xi). \tag{5}$$

Eq. (4) implies that, we obtain an extra control direction other than the original input vector-fields $g_1(\xi), g_2(\xi)$ in approximate sense, if $[g_1, g_2](\xi)$ is independent to $g_1(\xi), g_2(\xi)$ at any ξ.

Let us turn to consider the case of systems with drift (1). If we apply the same control (3) to the system (1), a straightforward computation tells us

$$\xi(4T) = \xi(0) + 4Tf(\xi(0)) + \frac{1}{2}(4T)^2 \left.\frac{\partial f}{\partial \xi}\right|_{\xi(0)} f(\xi(0))$$
$$+ T^2[g_1, g_2](\xi(0)) + 2T^2([g_1, f] + [g_2, f])(\xi(0)) + O(T^3).$$

The first three terms come from Taylor expansion of $f(\xi)$, while the next three terms are concerned with the Lie bracket. In the case of the system (1),

$$[g_1, f](\xi) = \begin{bmatrix} \bar{u}_2 \sin\psi \\ -\bar{u}_2 \cos\psi \\ 0 \end{bmatrix}, \quad [g_2, f](\xi) = \begin{bmatrix} -\bar{u}_1 \sin\psi \\ \bar{u}_1 \cos\psi \\ 0 \end{bmatrix},$$

$$[g_1, g_2](\xi) = \begin{bmatrix} \sin\psi \\ -\cos\psi \\ 0 \end{bmatrix},$$

which are all parallel to the direction perpendicular to the orientation of the vehicle. However, what really happens for $\xi(0) = 0$, $u_1(t) = 0.8\cos t, u_2(t) = 0$ is that the vehicle eventually proceeds in the direction parallel to the original orientation, as shown in Fig. 3. This gap makes us aware of fundamental limitation of Taylor-series and Lie-bracket based approach.

4. SOLUTION OF THE TWO-WHEELED VEHICLE WITH DRIFT USING BESSEL FUNCTIONS

4.1 Basic property of Bessel functions

For further analysis, we introduce basic properties of Bessel functions. *Bessel function of the first kind*, origi-