Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-13 (2015) 111–116

Stability and Consensus of Electrical Circuits via Structural Properties

Sofía Avila–Becerril * **Gerardo Espinosa-Pérez** *
Paul Fernández–Carrillo *

* *Facultad de Ingeniería, Universidad Nacional Autónoma de México, Edificio de Posgrado, 2o. piso, Ciudad Universitaria, 04510 México D.F., MEXICO (e-mail: soavbec@comunidad.unam.mx; gerardoe@unam.mx; paulfdz@outlook.com).*

Abstract: In this paper stability and consensus on electrical circuits is approached. The novelty of the presented results lies in the fact that, contrary to the usual practice of establishing these properties for a given circuit, generic features of this class of networks are interpreted in terms of interconnections of the circuit elements to conceive specific topologies for which both stability and consensus are guaranteed. Fundamental for this achievement is the Hamiltonian structure exhibited by the circuits, since the features enjoyed by this kind of dynamical systems allow to systematically state the structural (interconnection) properties under which stability is assured while conditions to conclude consensus are derived from the analysis of its *equilibria*.

Keywords: Electrical circuits, Hamiltonian systems, Consensus, Graph theory.

1. INTRODUCTION

Since the establishment of Kirchhoff's laws in the XIX century, the study of electrical circuits has lead to a very powerful theory that allows for a deep understanding of this kind of networks. This knowledge has been useful to recognize properties of the dynamical behavior under different scenarios, for example, linear (Desoer and Kuh (1969)) and nonlinear circuits (Brayton and Moser (1964)), including switched topologies (Rashid (2013)).

Among the vast number of possibilities to approach electrical circuits, one that has shown to be particularly useful is related with the Lagrangian and Hamiltonian descriptions of this kind of systems (Ortega et al. (1998)). The systematic derivation of mathematical models that exhibit an immediate physical interpretation, both for its structure and its behavior, has make of this perspective a solid bastion to develop new knowledge, for example control strategies, about these dynamical systems.

Actually, in the electrical circuits current literature it is possible to find results ranging from characterization (Van der Schaft and Maschke (2013)) to control (Ortega et al. (2003)) and applications that includes both small (Jeltsema and Scherpen (2003)) and large (Fiaz et al. (2013)) size systems.

One particular approach that can be used, besides to solve the controller design problem, to venture to the systems design field is the so-called *Control by Interconnection* (CbI) methodology (Ortega et al. (2008)). Roughly speaking, under this perspective the objective is to look at the controller as one dynamical system that interconnected with other dynamical system (the plant) generates a new dynamical system with desired properties. Concerning electrical circuits, some results are presented in (Van der Schaft (2010)) by exploiting the concept of open graphs.

Inspired by the CbI methodology design, the aim of this paper is to approach the study of electrical circuits, but instead of following the usual practice of finding dynamic properties of a given system, from the perspective of *imposing* a topological structure to the circuit that guarantees that those desired properties will be achieved. In some sense, the aim is to achieve a given control objective by designing the structure of the system.

The motivation of this study has its roots in the field of Electrical Power Systems (Kundur et al. (1994)) where both Voltage and Frequency stability problems are usually confronted by adding to the network new elements, called compensators, adequately located. In a general context and from a circuit viewpoint, this compensation action does not mean anything else that adding new capacitors, inductors and resistors in specific locations and in such a way that the desired behavior of the Power System is achieved.

The results presented in this contribution are focused to guarantee two properties for a given electrical circuit. First, the unavoidable stability property, and second, considering the aforementioned application, achievement of consensus for some variables of the network, in this case, the particular problem of consensus for the capacitor voltages is studied.

For the statement of the proposed results, concepts from Graph theory applied to electrical circuits are exploited. In this sense, generic properties for this kind of networks are used and, except for the case of resistors, nonlinear lumped one-port elements are considered.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.10.223

The rest of the paper is organized as follows: In Section 2 using well–known results from the Graph Theory (Bondy and Murty (1976)) the class of electrical circuits considered in the paper is characterized. Section 3 is devoted to the presentation of the dynamic model for the circuits followed, in Section 4, by its stability analysis. The main results of the paper are included in Section 5 where the Consensus problem is approached. Some concluding remarks are discussed in Section 6.

2. ELECTRICAL CIRCUIT GRAPHS AND PROBLEM FORMULATION

An electrical circuit can be viewed as a directed graph $\mathcal{G}$ consisting of a finite set of *nodes* $\mathcal{V}$ and a finite set of *edges* $\mathcal{E}$, together with a mapping from $\mathcal{E}$ to $\mathcal{V}$ such that to any edge $\epsilon \in \mathcal{E}$ corresponds an ordered pair $(v, w) \in \mathcal{V} \times \mathcal{V}$, with $v \neq w$. In the context of electrical circuits, the nodes are the usual interconnection points while the edges are lumped one-port (two-terminal) elements. Throughout the paper, it will be considered that there exist n nodes and b edges. Moreover, it will also be considered that the graph is connected in the sense that each node can be reached from any other node by tracing a path through the edges.

To each of the lumped one-port elements, represented by edges, are associated two variables, namely: the voltage e *across* its terminals and the current f that flows *through* it. Under these conditions voltage is an extensive variable, requiring a reference to be specified, while current is an intensive variable, which does not require a reference to be specified. Due to its association with these variables, the edge that connects two nodes (v, w) becomes an *oriented edge* for which (in this paper) it is adopted the convention that the orientation of the edge coincides with the direction of a positive current and a decreasing voltage, i.e. if the positive current flows from v to w, the voltage in tail node v is greater than the voltage of the head node w.

Once the lumped elements are interconnected, their port variables must satisfy constraints that are given by the celebrated Kirchhoff Current and Voltage Laws (KCL and KVL, respectively). In this paper, these constraints are stated in terms of *basic* cutsets and loopsets for a given *tree*, and its corresponding *co-tree*, of the graph (Wellstead (1979)). Remember that a tree is a connected sub-graph composed by the n nodes and $n - 1$ edges such that no loops are formed. Its complement, the $b - (n - 1)$ edges, conform the corresponding co-tree. The tree edges are called *branches* while the co-tree edges are *chords*. In addition, a basic cutset is a set of edges which elements are one branch and some or all the chords. A basic loopset is a set of one chord and some or all the branches connected by nodes in such a way that a closed loop is formed.

It is well-known that KCL is satisfied for every basic ambit and that KVL holds for every basic loop. In this sense, there exist $n - 1$ independent current constrains and $b - (n - 1)$ independent voltage constraints. If it is defined the b-dimensional vector space Λ_1 $(\mathbb{R}^b)$ as the space of currents through the edges and Λ^1 its dual space, composed by the voltages across the edges, and both $f \in \Lambda_1$ and $e \in \Lambda^1$ are ordered such that the tree branch variables appear first, i.e.

$$f = \begin{bmatrix} f_t \\ f_c \end{bmatrix} \in \Lambda_1; \quad e = \begin{bmatrix} e_t \\ e_c \end{bmatrix} \in \Lambda^1$$

with $f_t \in \mathbb{R}^{(n-1)}, e_t \in \mathbb{R}^{(n-1)}$ the currents and voltages of the branches and $f_c \in \mathbb{R}^{b-(n-1)}, e_c \in \mathbb{R}^{b-(n-1)}$ the currents and voltages of the chords, then the current constrains of the graph can be written as

$$[\, I \;\; H \,] \begin{bmatrix} f_t \\ f_c \end{bmatrix} = 0 \tag{1}$$

while the voltages constraints take the form

$$\left[\, -H^T \;\; I \,\right] \begin{bmatrix} e_t \\ e_c \end{bmatrix} = 0 \tag{2}$$

where I are identities matrices of proper dimensions and $H \in \mathbb{R}^{(n-1)\times b-(n-1)}$.

At this point it is important to state several properties about matrix H that will be fundamental for the results presented in this paper:

- One advantage of writing the circuit constraints as in (1) and (2) is that it can be clearly recognized that the branches currents can be generated as a linear combination, given by H, of the chords currents since
$$f_t = -H f_c \tag{3}$$
while the voltages chords are obtained as a linear combination of the voltages branches as can be viewed from
$$e_c = H^T e_t \tag{4}$$
- It must be noticed that matrix H is not the usual *node* reduced incidence matrix. Instead of, this matrix will be called *basic* incidence matrix since it is related with basic cutsets and loopsets.
- From an structural viewpoint, each row of matrix H indicates which of the lumped co-tree elements are incident on the same basic cutset than each of the branches. In correspondence, each column of this matrix indicates which of the lumped tree elements belong to the same basic loopset than each of the chords.

On the basis of the properties presented above, it is possible to formulate the problem approached in this paper as:

Given an electrical circuit graph impose, by interconnecting in an adequate way lumped elements, the structure of matrix $H \in \mathbb{R}^{(n-1)\times b-(n-1)}$ such that stability of the circuit and consensus of some port variables are achieved.

3. DYNAMIC BEHAVIOR OF ELECTRICAL CIRCUIT GRAPH

Up to this point, the topological structure of an electric circuit has been stated via the constraints (3) and (4). From these equations a dynamical model for the circuit is obtained by substituting the constitutive relations for each of the lumped elements. To do this, the usual three kind of elements must be considered, namely:

- Sources.