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Explicit second-order accurate method for the passive guaranteed simulation of port-Hamiltonian systems

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Abstract: This paper presents a method for the passive guaranteed simulation of a class of finite-dimensional nonlinear port-Hamiltonian systems. This method combines two processes to reach both the second order accuracy and explicit computations. First, we design a one-step two-stage implicit numerical method for Port-Hamiltonian systems that preserves passivity. Second, a change of state is proposed to yield an explicit computation. It requires assumptions on the Hamiltonian variations. The complete method is illustrated on two basic examples for which these assumptions are fulfilled.

Keywords: energy storage, port-Hamiltonian systems, nonlinear model, partial differential equations, passive simulation, consistency.

1. INTRODUCTION

Passive simulation is an important area of research as it guarantees the numerical stability for nonlinear systems.

Several methods are available to compute passive simulations, including for the sensitive issue of Hamiltonian mechanical systems which are conservative. They include Digital Waveguides, Wave Digital Filters and energy-conserving finite difference schemes (see Vilain (2010); Bilbao (2009); Julius (2010) for a review). In this work, we consider the port-Hamiltonian formulation which provides passive descriptions (decomposed into conservative and dissipative parts) of physical systems and of their combinations, in the continuous time domain Duindam (2009); Van der Schaft (2014).

Concerning Hamiltonian systems, numerous works have been done to compute passive simulation including for high-order accuracy Munthe-Kaas (1998); Del Buono (2002); Iserles (2000). For port-Hamiltonian systems, a numerical scheme based on a discrete definition of the Hamiltonian gradient has been described by Yalin (2015); Aoues (2014); Falaize (2014). In general, this method is not second-order accurate and is implicit. The computation of the implicit method needs Newton-Raphson type algorithm in which convergence issues may appear. Also, a high sampling frequency is needed to decrease the consistency error in the case of a one-order numerical scheme. Whatever, those limitations increase the computing time.

This paper presents a method for the passive guaranteed simulation of finite-dimensional nonlinear port-Hamiltonian systems. It combines two processes to reach both the second order accuracy and explicit computations: a one-step two-stage implicit numerical method for Port-Hamiltonian systems and a change of state to yield an explicit computation. This method is applied to a particular class of finite-dimensional port-Hamiltonian system and requires a strong assumption on the Hamiltonian variations for the implicit process.

In the first part of this paper, recalls on Port-Hamiltonian Systems and passive-guaranteed simulations are done including a study of the order of consistency. In a second part, we present the explicit second-order accurate method with two examples. Finally, results of simulation are described.

2. RECALLS ON PORT-HAMILTONIAN SYSTEMS AND PASSIVE-GUARANTEED SIMULATIONS

2.1 Port-Hamiltonian systems

This section introduces a class of finite dimensional port-Hamiltonian System. A general presentation can be found in Duindam (2009).

A port-Hamiltonian system of state x, input u and output y can be represented by the following differential equations :

$$\dot{\mathbf{x}} = (\mathbf{J}(\mathbf{x}) - \mathbf{R}(\mathbf{x}))\partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \quad (1)$$

$$\mathbf{y} = \mathbf{G}(\mathbf{x})^T \partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{D}(\mathbf{x})\mathbf{u} \quad (2)$$

where the positive definite function H denotes the energy of the system with respect to the state, where matrices $\mathbf{J}$ and $\mathbf{D}$ are skew-symmetric and where $\mathbf{R}$ is positive definite ($\mathbf{R} \geq 0$). This formulation guarantees the passivity as it naturally encodes the power balance. Indeed, as $\mathbf{J}$ and $\mathbf{D}$ are skew-symmetric, we have $\partial_{\mathbf{X}} H(\mathbf{x})^T \mathbf{J}(\mathbf{x}) \partial_{\mathbf{X}} H(\mathbf{x}) = 0$, $\mathbf{u}^T \mathbf{D}(\mathbf{x})\mathbf{u} = 0$ and then,

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$$\dot{H} = \underbrace{\partial_{\mathbf{X}} H^T \dot{\mathbf{x}}}_{\text{Energy variation}} = - \underbrace{\partial_{\mathbf{X}} H^T \mathbf{R} \partial_{\mathbf{X}} H}_{\substack{>0 \\ \text{Dissipated Power}}} + \underbrace{\mathbf{y}^T \mathbf{u}}_{\text{Entering Power}} \quad (3)$$

2.2 Discrete-gradient based numerical method

In this paper, we consider a zero-order hold for input $\mathbf{u}$ ($\mathbf{u}(t) = \mathbf{u}_n$ for $t \in [t_n, t_{n+1}[$) and a sampling period h.

In order to obtain the discrete-time formulation of a port-Hamiltonian conserving the property of passivity, we can use a discrete gradient based method already introduced in Yalin (2015); Aoues (2014); Falaize (2014).

The derivative of the state in eq. (1) is approximated by forward Euler as

$$\dot{\mathbf{x}} \cong \frac{\delta \mathbf{x}_n}{h} = \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{h}. \quad (4)$$

Also, to ensure the power balance eq. (3) in the discrete-time case, the discrete version of $\partial_{\mathbf{X}} H^T \dot{\mathbf{x}}$ must equal the energy variation between two steps:

$$\partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}_n)^T \delta \mathbf{x}_n = H(\mathbf{x}_n + \delta \mathbf{x}_n) - H(\mathbf{x}_n) \quad (5)$$

where $\partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}_n)$ denotes the discrete gradient of the Hamiltonian. For a N-dimensional system, the energy variation can be decomposed into a sum of N differences, as for example:

$$\begin{aligned} H(\mathbf{x} + \delta \mathbf{x}) - H(\mathbf{x}) = {} & H(\mathbf{x} + \delta \mathbf{x}) - H(x_1, x_2 + \delta x_2, ...) \\ & + H(x_1, x_2 + \delta x_2, ...) - H(x_1, x_2, x_3 + \delta x_3, ...) \\ & + \dots \\ & + H(x_1, ..., x_{N-1}, x_N + \delta x_N) - H(\mathbf{x}). \end{aligned} \quad (6)$$

This decomposition depends on the chosen increment order χ of δx and leads to a particular version of discrete Hamiltonian gradient that verifies eq. (5):

$$\lceil \partial^d_{\mathbf{X}} H(\mathbf{x}, \delta \mathbf{x}) \rceil_\chi = \begin{bmatrix} \frac{H(x_1 + \delta x_1, ...) - H(x_1, x_2 + \delta x_2, ...)}{\delta x_1} \\ \dots \\ \frac{H(x_1, ..., x_{N-1}, x_N + \delta x_N) - H(x_1, ..., x_{N-1}, x_N)}{\delta x_N} \end{bmatrix} \quad (7)$$

More precisely, one chooses a particular solution of eq. (5) as a definition of a discrete gradient in order to ensure a discrete version of the power balance eq. (3). This solution is based on a particular order χ which may be chosen advantageously according to the different cases. Also, one can define a symmetrized version of the discrete gradient that satisfies eq. (5) by averaging the solution for every path:

$$\partial^d_{\mathbf{X}} H(\mathbf{x}, \delta \mathbf{x}) = \frac{1}{N!} \sum_\chi \lceil \partial^d_{\mathbf{X}} H(\mathbf{x}, \delta \mathbf{x}) \rceil_\chi. \quad (8)$$

It appears that the definition of the discrete gradient depends on the choice of coordinates. However, even in this case, this method is still relevant. This property is used in the section (3.2). Finally, with (4) and (8) the discrete port-Hamiltonian system is described by:

$$\frac{\delta \mathbf{x}_n}{h} = (\mathbf{J}(\mathbf{x}_n) - \mathbf{R}(\mathbf{x}_n)) \partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}_n) + \mathbf{G}(\mathbf{x}_n) \mathbf{u}_n \quad (9)$$

$$\mathbf{y}_n = \mathbf{G}(\mathbf{x}_n)^T \partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}_n) + \mathbf{D}(\mathbf{x}_n) \mathbf{u}_n \quad (10)$$

where $\frac{\delta \mathbf{x}_n}{h} = \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{h} = \mathbf{s}_n$ and the slope $\mathbf{s}_n$ is this estimated state derivative. Thus, the discrete model leads to a one-step one-stage implicit and passive method for simulating:

$$\mathcal{M}_1 \begin{cases} \mathbf{s}_n &= (\mathbf{J}(\mathbf{x}_n) - \mathbf{R}(\mathbf{x}_n)) \partial^d_{\mathbf{X}} H(\mathbf{x}_n, h\mathbf{s}_n) + \mathbf{G}(\mathbf{x}_n) \mathbf{u}_n \\ \mathbf{x}_{n+1} &= \mathbf{x}_n + h\mathbf{s}_n \end{cases}$$

2.3 Numerical analysis

This section is devoted to study the consistency order of the above numerical method $\mathcal{M}_1$. In the following, we denote $\mathbf{S} = \mathbf{J} - \mathbf{R}$ and the subscript n is omitted to simplify the notations . The notation $f(h) = \circ(g(h))$ is equivalent to $\lim_{h \to 0} \frac{f(h)}{g(h)} = 0$.

As proved in Demailly (2006) for the general case, we can assert that the order of the method is at least ℓ if, and only if:

$$\partial^i_h \mathbf{s}\big|_{h=0} = \frac{1}{1+i} \partial^i_t (\mathbf{S}(\mathbf{x}) \partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \mathbf{u})$$

for $i \leq \ell - 1$.

Theorem 1. The numerical method $\mathcal{M}_1$ is consistent. (Order 1)

Proof. Considering that $H(\mathbf{x})$ is infinitely differentiable, we can write the Taylor series of the multi-variable Hamiltonian function:

$$H(\mathbf{x} + \delta \mathbf{x}) = H(\mathbf{x}) + \partial_{\mathbf{X}} H(\mathbf{x})^T \delta \mathbf{x} + \frac{1}{2} \delta \mathbf{x}^T \partial^2_{\mathbf{X}} H(\mathbf{x})^T \delta \mathbf{x} + \circ(|\delta \mathbf{x}|^2) \quad (11)$$

where $\partial^2_{\mathbf{X}} H$ is the Hessian matrix of the Hamiltonian. With (5) and (11), we have,

$$\partial^d_{\mathbf{X}} H(\mathbf{x}_n, \delta \mathbf{x}) = \partial_{\mathbf{X}} H(\mathbf{x}_n) + \frac{1}{2} \partial^2_{\mathbf{X}} H(\mathbf{x}_n) \delta \mathbf{x} + \circ(\delta \mathbf{x}). \quad (12)$$

It comes directly from ($\mathcal{M}_1$) and (12) that,

$$\mathbf{s}\big|_{h=0} = \mathbf{S}(\mathbf{x}) \partial^d_{\mathbf{X}} H(\mathbf{x}, 0) + \mathbf{G}(\mathbf{x}) \mathbf{u} \quad (13)$$

$$= \mathbf{S}(\mathbf{x}) \partial_{\mathbf{X}} H(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \mathbf{u} \quad (14)$$

This proves that the method is consistent.

Theorem 2. The numerical method $\mathcal{M}_1$ is second-order accurate if and only if $\mathbf{S} = \mathbf{J} - \mathbf{R}$ and $\mathbf{G}$ are independent of $\mathbf{x}$.

Proof. The order 1 proves that $\mathbf{s} = \dot{\mathbf{x}} + \circ(\mathbf{1})$. The equation (12) can be rewritten:

$$\partial^d_{\mathbf{X}} H(\mathbf{x}, \mathbf{s}h) = \partial_{\mathbf{X}} H(\mathbf{x}) + \frac{1}{2} \partial^2_{\mathbf{X}} H(\mathbf{x}) \mathbf{s} h + \circ(\mathbf{h})$$

Thus,

$$\partial_h \mathbf{s} = \mathbf{S}(\mathbf{x}) \partial_h (\partial_{\mathbf{X}} H(\mathbf{x}) + \frac{1}{2} \partial^2_{\mathbf{X}} H(\mathbf{x}) \mathbf{s} h + \circ(\mathbf{h})) \quad (15)$$

$$= \mathbf{S}(\mathbf{x}) (\frac{1}{2} \partial^2_{\mathbf{X}} H(\mathbf{x}) \mathbf{s} + \circ(\mathbf{1})) \quad (16)$$

and,

$$\partial_h \mathbf{s}\big|_{h=0} = \frac{1}{2} \mathbf{S}(\mathbf{x}) \partial^2_{\mathbf{X}} H(\mathbf{x}) \dot{\mathbf{x}} \quad (17)$$

$$= \frac{1}{2} \mathbf{S}(\mathbf{x}) \partial_t (\partial_{\mathbf{X}} H(\mathbf{x})). \quad (18)$$