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Zero dynamics for waves on networks ⋆

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Abstract: Consider a network with linear dynamics on the edges, and observation and control in the nodes. Assume that on the edges there is no damping, and so the dynamics can be described by an infinite-dimensional, port-Hamiltonian system. For general infinite-dimensional systems, the zero dynamics can be difficult to characterize and are sometimes ill-posed. However, for this class of systems the zero dynamics are shown to be well-defined. Using the underlying structure, simple characterizations and a constructive procedure can be obtained.

Keywords: Port-Hamiltonian system, distributed parameter systems, boundary control, zero dynamics, networks, coupled wave equations.

1. INTRODUCTION

The zeros of the transfer function of a system are well-known to be important to controller design for finite-dimensional systems; see for instance, the textbooks Doyle et al. (1992); Morris (2001). For example, the poles of a system controlled with a constant feedback gain move to the zeros of the open-loop system as the gain increases. Furthermore, regulation is only possible if the zeros of the system do not coincide with the poles of the signal to be tracked. Another example is sensitivity reduction - arbitrary reduction of sensitivity is only possible all the zeros are in the left-hand-plane. Right-hand-plane zeros restrict the achievable performance; see for example, Doyle et al. (1992) . The inverse of a system without right-hand-plane zeros can be approximated by a stable system, such systems are said to be minimum-phase.

The zero dynamics are a fundamental concept relating to the differential equation description. The zero dynamics are the dynamics of the system obtained by choosing the input u so that the output y is identically 0. This will only be possible for initial conditions in some subspace of the original subspace. For linear systems with ordinary differential equation models, the eigenvalues of the zero dynamics correspond to the zeros of the transfer function. Zero dynamics are well understood for finite-dimensional systems, and have been extended to nonlinear finite-dimensional systems Isidori (1999).

But many systems are modeled by delay or partial differential equations. This leads to an infinite-dimensional state space, and also an irrational transfer function. As for finite-dimensional systems, the zero dynamics are important. For instance, results on adaptive control and on high-gain feedback control of infinite-dimensional systems, see (Logemann and Owens, 1987; Logemann and Townley, 1997, 2003; Logemann and Zwart, 1992; Nikitin and Nikitina, 1999, e.g.), require the system to be minimum-phase. Moreover, the sensitivity of an infinite-dimensional minimum-phase system can be reduced to an arbitrarily small level and stabilizing controllers exist that achieve arbitrarily high gain or phase margin Foias et al. (1996).

The notion of minimum-phase can be extended to infinite-dimensional systems; see in particular Jacob et al. (2007) for a detailed study of conditions for second-order systems. Care needs to be taken since a system can have no right-hand-plane zeros and still fail to be minimum-phase. The simplest such example is a pure delay. There are a number of ways to define the zeros of a system; for systems with a finite-dimensional state-space all these definitions are equivalent. However, systems with delays, or partial differential equation models have state-space representations with an infinite-dimensional state space. Since the zeros are often not accurately calculated by numerical approximations Cheng and Morris (2003); Clark (1997); Grad and Morris (2003); Lindner et al. (1993) it is useful to obtain an understanding of their behaviour in the original infinite-dimensional context. Extensions from the finite-dimensional situation are complicated not only by the infinite-dimensional state-space but also by the unboundedness of the generator A.

In this paper, we consider zero dynamics of a class of partial differential equations with boundary control. For infinite-dimensional control systems where interchanging the role of the control and the output leads to a well-posed system, calculation of the zero dynamics is straightforward. Such systems must be non-strictly proper in a very strict sense, and this assumption is generally not satisfied. For strictly proper systems, the zero dynamics can only be calculated in special cases. For systems with bounded control and observation, the zero dynamics can calculated, although they are not always well-posed Zwart (1989); Morris and Rebarber (2007, 2010). In Byrnes et al.

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(1994) the zero dynamics are found for a class of parabolic systems defined on an interval with collocated boundary control and observation. However, no other results on zero dynamics for strictly proper systems with boundary control and observation are known. Here we consider an important class of these systems, port-Hamiltonian systems. Such models are derived using a variational approach and many situations of interest, in particular waves and vibrations, can be described in a port-Hamiltonian framework. In this paper it is assumed that the wave speeds are commensurate. For these systems, the zero dynamics are well-defined. Furthermore, the zero dynamics can be calculated using simple linear algebra calculations. This is illustrated with some examples.

2. PROBLEM FORMULATION

Consider systems of the form

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x(\zeta,t)), \quad \zeta \in (0,b), t \geq 0 \qquad (1)$$

$$u(t) = W_{B,1} \begin{bmatrix} x(b,t) \\ x(0,t) \end{bmatrix}, \quad t \geq 0 \qquad (2)$$

$$0 = W_{B,2} \begin{bmatrix} x(b,t) \\ x(0,t) \end{bmatrix}, \quad t \geq 0 \qquad (3)$$

$$y(t) = W_C \begin{bmatrix} x(b,t) \\ x(0,t) \end{bmatrix}, \quad t \geq 0, \qquad (4)$$

where P_1 is an Hermitian invertible $n \times n$-matrix, $\mathcal{H}$ is a positive $n\times n$-matrix, and $W_B := \left[\begin{smallmatrix} W_{B,1} \\ W_{B,2} \end{smallmatrix}\right]$ is a $n\times 2n$-matrix of rank n. Such systems are said to be *port-Hamiltonian,* see Le Gorrec et al. (2005); Villegas (2007); Jacob and Zwart (2012).

The matrices $P_1\mathcal{H}$ possess the same eigenvalues counted according to their multiplicity as the matrix $\mathcal{H}^{1/2}P_1\mathcal{H}^{1/2}$, and as $\mathcal{H}^{1/2}P_1\mathcal{H}^{1/2}$ is diagonalizable the matrix $P_1\mathcal{H}$ is diagonalizable as well. Moreover, zero is not an eigenvalue of $P_1\mathcal{H}$ and all eigenvalues are real, that is, there exists an invertible matrix S such that

$$P_1\mathcal{H} = S^{-1}\mathrm{diag}(p_1, \cdots, p_k, n_1, \cdots, n_l)S.$$

Here $p_1, \cdots, p_k > 0$ and $n_1, \cdots, n_l < 0$. We assume that the numbers $p_1, \cdots, p_k, -n_1, \cdots, -n_l$ are commensurate, that is, there exists a number $d \geq 0$ and $a_1, \cdots, a_k, b_1, \cdots, b_l \in \mathbb{N}$ such that

$$p_j = a_j d, \quad j = 1, \cdots, k, \qquad n_j = -b_j d, \quad j = 1, \cdots, l.$$

Introducing the new state vector

$$\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix} = Sx(\zeta,t), \qquad \zeta \in [0,b],$$

with $x_+(\zeta,t) \in \mathbb{C}^k$ and $x_-(\zeta,t) \in \mathbb{C}^l$, and writing

$$\mathrm{diag}(p_1, \cdots, p_k, n_1, \cdots, n_l) = \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix},$$

where Λ is a positive definite diagonal $k \times k$-matrix and Θ is a negative definite diagonal $l \times l$-matrix, the system (2)–(4) can be equivalently written as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix} = \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix}\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix}\right), \qquad (5)$$

$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}}_{K}\begin{bmatrix} \Lambda x_+(b,t) \\ \Theta x_-(0,t) \end{bmatrix} + \underbrace{\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}}_{Q}\begin{bmatrix} \Lambda x_+(0,t) \\ \Theta x_-(b,t) \end{bmatrix}, \quad (6)$$

$$y(t) = [O_{21} \ O_{22}]\begin{bmatrix} \Lambda x_+(b,t) \\ \Theta x_-(0,t) \end{bmatrix} + [R_{21} \ R_{22}]\begin{bmatrix} \Lambda x_+(0,t) \\ \Theta x_-(b,t) \end{bmatrix}, \quad (7)$$

where $t \geq 0$ and $\zeta \in (0,b)$.

Theorem 1. Zwart et al. (2010), (Jacob and Zwart, 2012, Thm. 13.2.2 and 13.3.1). The system (5)–(7) is well-posed on $L^2([0,b];\mathbb{C}^{n\times n})$ if and only if the matrix K is invertible.

Well-posedness implies that for every initial condition $x_0 \in L^2([0,b]];\mathbb{C}^n)$ and every input $u \in L^2_{\mathrm{loc}}((0,\infty);\mathbb{C}^p)$ the mild solution $\left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right]$ of the system (5)–(7) is well-defined in the state space $X := L^2([0,b];\mathbb{C}^n)$ and the output is well-defined in $L^2_{\mathrm{loc}}((0,\infty);\mathbb{C}^m)$. Moreover, for port-Hamiltonian systems, well-posedness implies that the system (5)–(7) is also regular, see Zwart et al. (2010) or (Jacob and Zwart, 2012, Section 13.3). Writing $[O_{21} \ O_{22}]\, K^{-1} = [* \ E]$ with $E \in \mathbb{C}^{m\times p}$, the matrix E equals the feedthrough operator of the system, see (Jacob and Zwart, 2012, Section 13.3). For the remainder of this paper it is assumed that K is invertible.

Definition 2. Consider the system (5)–(7) on the state space $X = L^2([0,b];\mathbb{C}^n)$. The largest output nulling subspace is

$$V^* = \{x_0 \in X \mid \text{ there exists a } u \in L^2_{\mathrm{loc}}((0,\infty);\mathbb{C}^p) : \text{the mild solution of (5)–(7) satisfies } y = 0\}$$

The zero dynamics is described by the system

$$\frac{\partial}{\partial t}\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix} = \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix}\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix}\right), \qquad (8)$$

$$0 = \begin{bmatrix} K_{11} & K_{12} \\ O_{21} & O_{22} \end{bmatrix}\begin{bmatrix} \Lambda x_+(b,t) \\ \Theta x_-(0,t) \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} \Lambda x_+(0,t) \\ \Theta x_-(b,t) \end{bmatrix}, \quad (9)$$

$$u(t) = [K_{21} \ K_{22}]\begin{bmatrix} \Lambda x_+(b,t) \\ \Theta x_-(0,t) \end{bmatrix} + [Q_{21} \ Q_{22}]\begin{bmatrix} \Lambda x_+(0,t) \\ \Theta x_-(b,t) \end{bmatrix}, \quad (10)$$

where $t \geq 0$ and $\zeta \in (0,b)$.

3. INVERTIBLE FEEDTHROUGH OPERATOR

Inspection of (8)–(10) reveals that the largest output-nulling subspace $V^* = L^2([0,b];\mathbb{C}^n)$ has well-posed zero dynamics if and only

$$\tilde{K} := \left[\begin{smallmatrix} K_{11} & K_{12} \\ O_{21} & O_{22} \end{smallmatrix}\right]$$

is invertible (Theorem 1). In this case, the zero dynamics are well-posed on the entire state space.

Theorem 3. Assume that the number of inputs equals the number of outputs. Then the zero dynamics are well-posed on the entire state space if and only if the feedthrough operator of the original system is invertible.

Proof: In Section 2 we showed that the feedthrough operator E is given as

$$[O_{21} \ O_{22}]\, K^{-1} = [* \ E].$$

Hence if $u \neq 0$ lies in the kernel of E, then

$$[O_{21} \ O_{22}]\, K^{-1}\begin{bmatrix} 0 \\ u \end{bmatrix} = 0.$$