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The biased balance: Observation, formalism and interpretation of a dissymmetric measuring device

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ABSTRACT

This paper studies a balance whose unobservable fulcrum is not necessarily located at the middle of its two pans. It presents three different models, showing how this lack of symmetry modifies the observation, the formalism and the interpretation of such a biased measuring device. It argues that the biased balance can be an interesting source of inspiration for our abstract understanding of how a measuring device influences the measurement process.

Then, at last, as they were nearing the fountains for the fourth time, the father of all balanced his golden scales and placed a doom in each of them, one for Achilles and the other for Hector. As he held the scales by the middle, the doom of Hector fell down deep into the house of Hades-and then Phoebus Apollo left him.
Homer, *Iliad* XXII.

Give me a place to stand on, and I can move the earth.
Archimedes

1. Introduction

What would have happened had Apollo not taken his scales by the middle? Depending on what we assume to observe with such a biased measuring device, how can we formalize empirical observation and how can we interpret the numbers issued from measurement? This paper proposes a rigorous study of these questions in the context of a scale, or balance, that is not necessarily composed of arms of equal lengths.

A main motivation for broadening our understanding of measurement with the study of a biased balance lies in the universality of the unbiased balance for measurement and judgment. Osiris uses a balance to measure the soul of the dead in ancient Egypt. In the Greek epic tradition, deities like Apollo use a balance to decide of the fate of heroes. As a measuring device, it is discussed by Plato, Aristotle, Euclid and Archimedes [11,24]. It appears in the Bible as a symbol for rigor and exactness and in the Koran as a symbol of supreme wisdom. It symbolizes the invariable middle in ancient China, is part of the Sanskrit mythology and of the Indian and Tibetan spiritual traditions [5]. In the middle ages, the balance was essential to evaluate the price of goods and to allow for the development of trade [13]. Nowadays, it is a symbol of justice all over the modern world. It is ubiquitous in the philosophy of science [3,8,4] and is a seminal example for the foundations of measurement (e.g. [9,27,25]). Historically, the equal-arm balance has been a model for the measurement of objects and for the intuition of unbiased judgment. By studying a biased balance, we intend to better understand how measurement is affected by a biased measuring device and how biased judgements may be modelled.

This is especially true for the representational theory of measurement [9]. This abstract approach to the foundations of measurement formulates formal axioms that can describe empirical observation and be necessary and sufficient to prove the existence and uniqueness of a measuring scale. Let us show how this works with an equal-arm balance. Suppose we position an object, denoted *x*, on one of its pan and an object *y* on the other pan. Suppose that we observe that *x is lower than y*. This observation is formally described with a binary relation $\succ_0$ as "$x \succ_0 y$". Adding another object *z* to *x*,we observe that *x with z are lower than y*. Since this happens for any object *z*, the empirical regularity of such an observation leads to assume the following property:

$$\text{for all } x,y,z\text{:}\ \ x \succ_0 y \Rightarrow (x \circ z) \succ_0 y,$$

where "∘" naturally means the operation of jointly positioning two objects on the same pan of the balance. Further axioms then reflect the laws or regularities that can be observed, including in particular the

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following *additive independence* property:

for all x,y,z: $x \succ_0 y \Leftrightarrow (x \circ z) \succ_0 (y \circ z)$.

With sufficient axioms characterizing such an abstract and idealized setting (the measurement is performed in a locally uniform gravitational field, there is no uncertainty nor any other influence on the measuring process, etc.), the task of representational measurement is then to prove the existence of a function, say φ, which assigns a number to each object such that an object is lower than another on the balance if and only if it is assigned a greater number. Formally, we prove that there exists a real-valued function φ such that

$$x \succ_0 y \Leftrightarrow \varphi(x) > \varphi(y),$$

$$\varphi(x \circ y) = \varphi(x) + \varphi(y).$$

Such a representation theorem builds on Hölder's theorem (see [23] for an English translation) and the theory of extensive measurement (see [9, Chapter 3]). In such an abstract and idealized setting and because the balance is assumed to be of equal arms, the number $\varphi(x)$ can be interpreted as the mass of x as in classical mechanics. The function φ is unique up to multiplication by a positive constant and is called a ratio-scale (see [31]). Systematic predictions can be justified by this formalization. For instance, if the sum of the mass of y and the mass of z is greater than the mass of x, we predict with certainty that we will observe that *y with z is lower than x*. With this abstract and idealized model of the equal-arm balance (including the assumptions of a locally uniform gravitational field, etc.), the observed relation between objects does not depend on the measuring device and its formalization does not depend on the observer. Also, the observed empirical relation is formalized with formal (non-numerical) statements which univocally correspond with observation. Finally, a numerical representation is provided which measures objects and the function alone suffices to this measurement. Things are different with the biased balance. Depending on what we observe, the bias may induce less empirical regularities that must be reflected with weaker, and thus more general, axioms. A first question arises as whether we can still measure objects with a ratio-scale. Another question is whether we can measure the bias of the balance and if yes, what does that measure means.

A biased balance is a two-arm balance whose fulcrum is not necessarily located at the middle of the two pans. The principle of the balance with unequal arms as a measurement of torque has long been understood, at least since Archimedes' proof of the principle of the lever (Propositions 6 and 7 of Book I of *On the equilibrium of the planes*, see [11, p. 192]). Also, the so-called Roman or Steelyard balance, where objects positioned on a tray at one end of the beam are balanced by moving a counterweight along the opposite side of the beam, has been employed to weigh large bodies from the earliest time. Not only the principle of the lever had to be invoked, but also the account of the weight of the tray (or hook) used to hold the object to be weighed, which induces some complications (see for instance the *Liber de Canonio* in [24]). As shown in Suppes [32], these earlier mathematical approaches are very close to the contemporaneous theory of conjoint measurement [9, Chapter 6]. What they share in particular is that they start with two quantities (here weights and distances) which can be manipulated independently in order to observe their conjoint effect. In particular, it is assumed possible to select the distances from the fulcrum so that they are of appropriate proportions. Also, it is assumed that distances can be divided into segments of equal length. In this manner, we can use the device to measure torque and from the measurement of distance derive an indirect measurement of weights.

Our study of the biased balance is of interest and novelty because it does not assume that the distance from the fulcrum is an observable primitive. Depending on what we observe as a relation among objects, we characterize the implicit role of the bias. Hence, we do not start from two quantities playing similar roles but with one that is observable (the objects positioned on the balance and acting on it with their mass) and infer the role of a factor that is not directly observable (the bias of the balance). Because of the hidden role of the bias, the relation between objects presents less regularity. Therefore, we need to relax some of the properties of the axioms that are supposed to describe the empirical observation of an equal-arm balance. The biased balance being a form of generalization of the equal-arm balance (that the fulcrum is located in the middle is a special case), it provides a model as to how the representational theory of measurement can be generalized. For instance, the representational theory of measurement treats axioms such as completeness and transitivity as necessary for the existence of a ratio-scale.[1] The theory of biased measurement [15–17,14] shows how we can derive the existence of a ratio-scale while relaxing such axioms.

In order to study different assumptions about what can be observed and different formalizations of empirical observation, this paper presents three models of the biased balance. Each model assumes different ways to observe the behavior of the biased balance. Thus, each model shows distinct empirical regularities which are reflected in different set of axioms. Each set of axioms leads to a representation theorem proving that, even with the irregularities emanating from the bias of the balance, a ratio-scale measure of the mass of objects can be shown to exist. These theorems also reveal a numerical factor which quantifies these irregularities and which intuitively corresponds to the bias of the balance. The interpretation of such number is not necessarily obvious, and we make precise what it means and what it quantifies. The biased balance hence leads to a more detailed analysis of the correspondence between empirical observation and its formalization as a relational structure. This step is usually taken for granted in the theory of representational measurement, due to an implicit assumption of the symmetry of the measuring device. Methodologically, we study the biased balance following three fundamental questions:

1. What do we observe and how can we formally describe it?
2. What numerical representation can be constructed from this formal description?
3. What is the meaning of the numbers that we have constructed?

These questions are essential to a clear and precise understanding of the use of numbers and of mathematical models in sciences. Because the biased balance shows how the theory of representational measurement may be broadened to apply to phenomena which do not present the typical empirical regularities assumed by the symmetry of the measuring device, it contributes to address one of its most interesting critiques (e.g. [28,20,21,2]).

The rest of the paper is structured as follows. In Section 2, we present the basic terms and formal properties that we use to study a biased balance. We also introduce the 3 models. In Section 3, we present the first model which assumes that we can observe on which arm of the balance each object is positioned. This model is the closest to the intuition that the biased balance measures the torque and that a conjoint approach should allow to measure both the mass and the distance that compose it. This is carried out by formally defining "extended objects" that are composed of an object together with the arm on which it lies. In that case, we show how we can conjointly construct a ratio-scale that measures the mass of objects and, for each biased balance, a unique pair of numbers that measures the distances between the fulcrum and each pan. In Section 4, a second model assumes that we can observe whether a given object is positioned on the left or on the right from the observer perspective. We do not however define "extended objects" and let the formalism implicitly reflect the left and right distinction that depends on the observer. We show that this corresponds to the most general mathematical properties but still allows for a ratio-scale measuring objects to be constructed. Further, we show that we can

[1] Mathematically, any representation of the form $x \succsim y \Leftrightarrow \varphi(x) \geqslant \varphi(y)$ must assume that the relation $\succsim$ is complete and transitive.