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On the use of algebraic geometry for the design of high-gain observers for continuous-time polynomial systems

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Abstract: The goal of this paper is to apply some concepts and techniques from algebraic geometry to study the observability of nonlinear continuous-time polynomial systems. After deriving some new results for observability and embeddings, it is shown how to use such concepts to easily design high-gain observers. The proposed technique is illustrated by an application to the well known Rossler oscillator.

Keywords: Algebraic geometry, observability, high-gain observers, Rössler oscillator.

1. INTRODUCTION

One of the classical problems in control theory is that of designing state observers to obtain in real time estimates of unmeasurable state variables, using the available measured outputs. This has motivated the study of the structural property of observability and of various techniques for designing state observers, as in the works by Krener and Respondek (1985); Isidori (1995); Nijmeijer and van der Schaft (1990); Tsinias (1990); Kazantzis and Kravaris (1998); Menini and Tornambe (2002a,b, 2010a,b, 2011a).

Some of the works closer to the problems considered here are by Sussmann (1978); Sontag (1979); Takens (1981); Aeyels (1981a,b); Jouan and Gauthier (1996). In some of such works the problem of observability through sampled measurements is considered. In particular, an application using a CAS (Computer Algebra System) of the developments by Sontag (1979) was done by Nešić (1998).

When dealing with polynomial systems, it is very useful to use techniques from Algebraic Geometry, which provide the natural tools to state clear and easily testable results. This has been recognized by a variety of authors Diop (1991); Helmke et al. (2003); Tibken (2004); Menini and Tornambe (2014); Kawano and Ohtsuka (2013); Menini and Tornambe (2013a,b). The main novelty of this paper, with respect to such works, is that here observability and embedding problems are studied with the focus of observer design, whence results based on algebraic geometry are stated that allow the easy writing of explicit forms for an inverse of the observability map and of an embedding of the given system. Although we illustrate the application of our techniques with respect to high gain observers (see, e.g.,Tornambe (1989); Esfandiari and Khalil (1992); Gauthier and Kupka (2001); Hammouri et al. (2010)), it is stressed that, in principle, they can be applied to other ways of designing observers.

2. PRELIMINARIES AND NOTATION

The goal of this section is to briefly resume the basic notions of algebraic geometry (Cox et al. (1998, 2007)) that will be used in the sequel.

Let $x \in \mathbb{R}^n$, $x = [\, x_1 \; ... \; x_n \,]^\top$, let $\mathbb{R}[x_1, ..., x_n]$ (or, shortly, $\mathbb{R}[x]$) be the commutative ring of all scalar polynomials in $x_1, ..., x_n$, and let $\mathbb{R}(x_1, ..., x_n)$ (or, shortly, $\mathbb{R}(x)$) be the field of all scalar rational functions of $x_1, ..., x_n$; $\mathbb{R}^n[x]$ denotes the set of vector functions with n entries in $\mathbb{R}[x]$.

Let $p_1, ..., p_m$ be polynomials in $\mathbb{R}[x]$; the set

$$\mathbf{V}_n(p_1, ..., p_m) := \{x \in \mathbb{R}^n : p_i(x) = 0 \text{ for all } i = 1, ..., m\}$$

is called the *affine variety* [1] (briefly, *variety*) of $\mathbb{R}^n$ defined by $p_1, ..., p_m$. As an example, $\mathbf{V}_1(x_1)$ is a point of $\mathbb{R}^1$, $\mathbf{V}_2(x_1)$ is a line of $\mathbb{R}^2$, and so on. If $\mathcal{V}_a = \mathbf{V}_n(p_1, ..., p_{m_a})$ and $\mathcal{V}_b = \mathbf{V}_n(q_1, ..., q_{m_b})$, for some polynomials p_i and q_j in $\mathbb{R}[x]$, then both $\mathcal{V}_a \cap \mathcal{V}_b$ and $\mathcal{V}_a \cup \mathcal{V}_b$ are varieties, and they are given by:

$$\begin{aligned}\mathcal{V}_a \cap \mathcal{V}_b &= \mathbf{V}_n(p_1, ..., p_{m_a}, q_1, ..., q_{m_b}),\\ \mathcal{V}_a \cup \mathcal{V}_b &= \mathbf{V}_n(p_1 q_1, ..., p_{m_a} q_1, ..., p_1 q_{m_b}, ..., p_{m_a} q_{m_b}).\end{aligned}$$

A subset $\mathcal{I}$ of $\mathbb{R}[x]$ is a *polynomial ideal* (briefly, an *ideal*) if it satisfies:

(1) if $p, q \in \mathcal{I}$, then $p + q \in \mathcal{I}$;

(2) if $p \in \mathcal{I}$ and $q \in \mathbb{R}[x]$, then $pq \in \mathcal{I}$.

Let $p_1, ..., p_m$ be polynomials in $\mathbb{R}[x]$; it is possible to show that the set

$$\langle p_1, ..., p_m \rangle_n := \{q_1 p_1 + ... + q_m p_m : q_i \in \mathbb{R}[x], i = 1, ..., m\}$$

is an ideal of $\mathbb{R}[x]$ and, in particular, it is called the *ideal generated* by $p_1, ..., p_m$.

[1] In some texts $\mathbf{V}_n(p_1, ..., p_m)$ is called *algebraic set*, whereas the term affine variety is reserved for the case when $\mathbf{V}_n(p_1, ..., p_m)$ is irreducible.

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An ideal $\mathcal{I}$ is said to be finitely *generated* if there exist polynomials $p_1, ..., p_m$ in $\mathbb{R}[x]$ such that $\mathcal{I} = \langle p_1, ..., p_m \rangle_n$; the set of the ideal *generators* $\{p_1, ..., p_m\}$ is called a *basis* of $\mathcal{I}$. By the Hilbert Basis Theorem (Cox et al., 2007), every ideal of $\mathbb{R}[x]$ is finitely generated. Clearly, if $\langle p_1, ..., p_{m_a} \rangle_n = \langle q_1, ..., q_{m_b} \rangle_n$, for some polynomials p_i and q_j in $\mathbb{R}[x]$, then $\mathbf{V}_n(p_1, ..., p_{m_a}) = \mathbf{V}_n(q_1, ..., q_{m_b})$.

Let $\mathcal{V} = \mathbf{V}_n(p_1, ..., p_m)$ be the variety defined by $p_1, ..., p_m$ in $\mathbb{R}[x]$. By definition, polynomials $p_1, ..., p_m$ vanish on $\mathcal{V}$, whence all polynomials belonging to $\langle p_1, ..., p_m \rangle_n$ vanish on $\mathcal{V}$. Note that there may be polynomials $p \notin \langle p_1, ..., p_m \rangle_n$ that vanish on $\mathcal{V}$; this renders necessary the following definitions.

Let $\mathcal{V}$ be a variety of $\mathbb{R}^n$; it is possible to show that the set

$$\mathbf{I}_n(\mathcal{V}) := \{p \in \mathbb{R}[x_1, ..., x_n] : p(x) = 0 \text{ for all } x \in \mathcal{V}\} \quad (1)$$

is an ideal and, in particular, it is called the *ideal of* $\mathcal{V}$; note that the empty set $\emptyset$ is a variety and that $\mathbf{I}_n(\emptyset) = \langle 1 \rangle_n = \mathbb{R}[x]$. Clearly, $\langle p_1, ..., p_m \rangle_n \subseteq \mathbf{I}_n(\mathbf{V}_n(p_1, ..., p_m))$, and the equality $\langle p_1, ..., p_m \rangle_n = \mathbf{I}_n(\mathbf{V}_n(p_1, ..., p_m))$ need not occur, e.g., if $n = 1$, all the polynomials $x_1^2 q(x_1)$ of $\langle x_1^2 \rangle_1$, with $q \in \mathbb{R}[x_1]$, vanish on the variety $\mathcal{V} = \mathbf{V}_1(x_1^2) = \{0\}$, but $\mathbf{I}_1(\mathbf{V}_1(x_1^2)) = \langle x_1 \rangle_1$ and $\langle x_1^2 \rangle_1 \neq \langle x_1 \rangle_1$ since $p(x_1) = x_1$ vanishes on $\{0\}$, but does not belong to $\langle x_1^2 \rangle_1$.

Let $\mathcal{V}_a$ and $\mathcal{V}_b$ be two varieties of $\mathbb{R}^n$, then

(1) $\mathcal{V}_a \subset \mathcal{V}_b$ if and only if $\mathbf{I}_n(\mathcal{V}_a) \supset \mathbf{I}_n(\mathcal{V}_b)$;

(2) $\mathcal{V}_a = \mathcal{V}_b$ if and only if $\mathbf{I}_n(\mathcal{V}_a) = \mathbf{I}_n(\mathcal{V}_b)$;

this, in particular, shows that the map $\mathbf{I}_n(\mathcal{V})$ is one-to-one.

For instance if $n = 1$, $\mathcal{V}_a = \{1\}$ and $\mathcal{V}_b = \{1, 2\}$, then $\mathbf{I}_1(\mathcal{V}_a) = \langle x_1 - 1 \rangle_1$ and $\mathbf{I}_1(\mathcal{V}_b) = \langle (x_1 - 1)(x_1 - 2) \rangle_1$, with $\mathbf{I}_1(\mathcal{V}_a) \supset \mathbf{I}_1(\mathcal{V}_b)$.

Let $\mathcal{I}$ be an ideal of $\mathbb{R}[x]$; it is possible to show that the set

$$\mathbf{V}_n(\mathcal{I}) := \{x \in \mathbb{R}^n : p(x) = 0 \text{ for all } p \in \mathcal{I}\} \quad (2)$$

is a variety and that $\mathbf{V}_n(\mathcal{I}) = \mathbf{V}_n(p_1, ..., p_m)$ for any basis $\{p_1, ..., p_m\}$ of $\mathcal{I}$.

A variety $\mathcal{V}$ can be studied through the corresponding ideal $\mathbf{I}_n(\mathcal{V})$ given by (1) and, conversely, an ideal $\mathcal{I}$ can be studied through the corresponding variety $\mathbf{V}_n(\mathcal{I})$ given by (2), but $\mathbf{I}_n(\mathbf{V}_n(\mathcal{I}))$ need not coincide with $\mathcal{I}$ (e.g., $\mathbf{I}_1(\mathbf{V}_1(\langle x_1^2 \rangle_1)) = \langle x_1 \rangle_1$), whence the map $\mathbf{V}_n(\mathcal{I})$ is not one-to-one (e.g., $\mathbf{V}_1(\langle 1 \rangle_1) = \mathbf{V}_1(\langle 1 + x_1^2 \rangle)$ and $\langle 1 \rangle_1 \neq \langle 1 + x_1^2 \rangle_1$).

Given any subset $\mathcal{S}$ of $\mathbb{R}^n$, it is easy to check that

$$\mathbf{I}_n(\mathcal{S}) := \{p \in \mathbb{R}[x_1, ..., x_n] : p(x) = 0 \text{ for all } x \in \mathcal{S}\}$$

is an ideal even if $\mathcal{S}$ is not a variety; hence, $\mathbf{V}_n(\mathbf{I}_n(\mathcal{S}))$ is a variety that contains $\mathcal{S}$. In particular, it is the smallest variety containing $\mathcal{S}$; $\mathbf{V}_n(\mathbf{I}_n(\mathcal{S}))$ is called the *Zariski closure* of $\mathcal{S}$. As an example that will be useful in the following, if $\mathcal{S} \subseteq \mathbb{R}^n$ is open, $\mathcal{S} \neq \emptyset$, then $\mathbf{I}_n(\mathcal{S}) = \{0\}$ and its Zariski closure coincides with the whole $\mathbb{R}^n$ ($\mathbf{V}_n(\mathbf{I}_n(\mathcal{S})) = \mathbb{R}^n$).

Some polynomials $q_1, ..., q_m \in \mathbb{R}[x]$ are *algebraically dependent* if there exists a non-zero polynomial $p \in \mathbb{R}[q]$, where $q = [\, q_1 \ \dots \ q_m \,]^\top$, such that $p(q_1(x), ..., q_m(x)) = 0$, $\forall x \in \mathbb{R}^n$ (*algebraically independent*, otherwise). Necessary and sufficient condition for $q_1, ..., q_m$ to be algebraically dependent is that $\operatorname{rank}_{\mathbb{R}(x)} \left(\frac{\partial q}{\partial x} \right) < m$.

Fix a total ordering $>$ of the monomials of $\mathbb{R}[x]$. For any $p \in \mathbb{R}[x]$, with $p \neq 0$, one can write

$$p(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + ... + a_\ell x^{\alpha_\ell},$$

where $a_i \in \mathbb{R}$, α_i is a multi-index, $i = 1, 2, ..., \ell$, and $x^{\alpha_1} > x^{\alpha_2} > ... > x^{\alpha_\ell}$; this allows one to define the *leading monomial* $\mathrm{LM}(p) = x^{\alpha_1}$ and the *leading coefficient* $\mathrm{LC}(p) = a_1$ of p.

Let $\{p_1, ..., p_m\}$ be a basis of an ideal $\mathcal{I}$ of $\mathbb{R}[x]$. A polynomial $r \in \mathbb{R}[x]$ is said to be *reduced* with respect to $\{p_1, ..., p_m\}$ if either $r = 0$ or no monomial that appears in r is divisible by $\mathrm{LM}(p_i), i = 1, ..., m$. A polynomial $r \in \mathbb{R}[x]$, which is reduced with respect to $\{p_1, ..., p_m\}$, is called a *remainder of the division of* $p \in \mathbb{R}[x]$ *by* $\{p_1, ..., p_m\}$ if $p - r \in \langle p_1, ..., p_m \rangle_n$.

Example 1. Fix the deglex (degree lexicographic) monomial ordering on $\mathbb{R}[x_1, x_2]$ with $x_2 > x_1$. Let $p(x) = x_1 x_2^2 - x_1$ and $\langle p_1, p_2 \rangle_2 = \langle x_1 x_2 - x_2, x_2^2 - x_1 \rangle_2$. It is easy to see that $r_1 = 0$ and $r_2 = x_1^2 - x_1$ are both reduced with respect to $\{p_1, p_2\}$ and that $p = x_2 p_1 + p_2 + r_1$ and $p = 0 p_1 + x_1 p_2 + r_2$, which show that the remainder of the division by an arbitrary basis need not be unique.

Let $\{g_1, ..., g_m\}$ be a set of non-zero polynomials in $\mathbb{R}[x]$. A basis $\{g_1, ..., g_m\}$ of an ideal $\mathcal{I}$ of $\mathbb{R}[x]$ is called a *Gröbner basis* of $\mathcal{I}$ if, for any $p \in \mathbb{R}[x]$, the remainder of the division of p by $\{g_1, ..., g_m\}$ is unique. Every non-zero ideal $\mathcal{I}$ of $\mathbb{R}[x]$ has a Gröbner basis, which need not be unique [2].

A monomial ordering on $\mathbb{R}[x_a, x_b]$, with $x_a \in \mathbb{R}^{n_a}$ and $x_b \in \mathbb{R}^{n_b}$, $n = n_a + n_b$, *eliminates* x_a if

$$x_a^\alpha > x_a^\beta \Rightarrow x_a^\alpha x_b^\gamma > x_a^\beta x_b^\delta,$$

for all multi-indices α, β for which $x_a^\alpha > x_a^\beta$ and for all multi-indices γ, δ. For instance, the lexicographic ordering with $x_1 > x_2 > ... > x_n$ eliminates $x_1, ..., x_\ell$, for all $1 \leq \ell < n$. Let $\mathcal{I}$ be an ideal of $\mathbb{R}[x_a, x_b]$. The elimination ideal of $\mathcal{I}$ that eliminates x_a is $\mathcal{I} \cap \mathbb{R}[x_b]$. Let $\{g_1, ..., g_m\}$ be a Gröbner basis for a monomial ordering that eliminates x_a. Then, the set obtained from $\{g_1, ..., g_m\}$ by retaining only the elements that do not depend on x_a (i.e., $\{g_1, ..., g_m\} \cap \mathbb{R}[x_b]$) is a Gröbner basis of the elimination ideal $\mathcal{I} \cap \mathbb{R}[x_b]$ for the monomial ordering on $\mathbb{R}[x_b]$ induced by $>$.

Example 2. Let $\mathcal{I} = \langle x_1 + x_2^2, x_1 - x_2 \rangle_2$. Fix the lexicographic ordering with $x_1 > x_2$, which eliminates x_1; a Gröbner basis of $\mathcal{I}$ for such an ordering is $\{x_2^2 + x_2, x_1 - x_2\}$, whence the elimination ideal of $\mathcal{I}$ that eliminates x_1 is $\mathcal{I} \cap \mathbb{R}[x_2] = \langle x_2^2 + x_2 \rangle_1$; in particular, $\{x_2^2 + x_2\}$ is a Gröbner basis of $\mathcal{I} \cap \mathbb{R}[x_2]$ for the induced ordering (in this case trivial).

3. OBSERVABILITY AND EMBEDDINGS OF POLYNOMIAL SYSTEMS

Consider the following polynomial system

$$\dot{x} = f(x), \quad (3a)$$

[2] By using the stronger notion of *reduced* Gröbner basis unicity is achieved, but this is not needed here.