Proceedings of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

On Networked Evolutionary Games Part 1: Formulation ⋆

Hongsheng Qi * Daizhan Cheng * Hairong Dong **

* Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P.R.China (e-mail: qihongsh@amss.ac.cn, dcheng@iss.ac.cn)
** Beijing Jiaotong University, Beijing 100044, P.R.China (e-mail: hrdong@bju.edu.cn)

Abstract: This paper presents a comprehensive modeling technique for networked evolutionary games (NEG). Three kinds of network graphs are considered, which are (i) undirected graph for symmetric games; (ii) directed graph for asymmetric games, and (iii) d-directed graph for symmetric games with partial neighborhood information. Three kinds of fundamental evolutionary games (FEGs) are discussed, which are (i) two strategies and symmetric (S-2); (ii) two strategies and asymmetric (A-2); and (iii) three strategies and symmetric (S-3). Three strategy updating rules (SUR) are introduced, which are (i) Unconditional Imitation (UI); (ii) Fermi Rule (FR); (iii) Myopic Best Response Adjustment Rule (MBRA). Then we review the fundamental evolutionary equation (FEE), and give the detailed formulation for different models. Finally, the network profile dynamics (NPD) of NEGs are investigated via their FEE.

Keywords: Networked evolutionary game, fundamental evolutionary equation, network profile dynamics, semi-tensor product of matrices

1. INTRODUCTION

In the last four decades or so, the investigation of evolutionary games (EG) has attracted a great attention from scientists in cross disciplines, because evolutionary game has wide background from biological systems (Taylor & Jonker, 1978; Charnov, 1982), economical systems (Sugden, 1986), social systems (Ohtsuki et al., 2006), physical systems (Nowak & May, 1992), *etc.*

In recent researches, the topological relationship among players of an EG is mostly ignored. That is, assume each player gambles with all others. In many practical cases the situation is not like this. Therefore, in recent years the networked EG (NEG) becomes a hot topic. Roughly speaking, an NEG adds a graph with players as its nodes and sides describing the neighborhoods of each players. Then each player only gambles with its neighbors (Nowak & May, 1992; Szabo & Toke, 1998; Santos et al., 2008). Since there are no many proper tools to deal with NEG, most of the researches are based on either simulations or statistics.

Recently, the semi-tensor product (STP) has been proposed for investigating (Boolean and k-valued logical) networks and network-based games (Cheng et al., 2011, 2012a). There are many other interesting developments such as (i) topological structure of notworks (Fornasini & Valcher, 2013b; Hochma et al., 2013); (ii) controllability and control design of various kinds of control networks (Laschov & Margaliot, 2012; Li & Sun, 2011a; Zhang & Zhang, 2013); (iii) optimal control and game related optimization (Laschov & Margaliot, 2012); (iv) network stability and stabilization (Li et al., 2013b); (v) technique for reducing complexity (Zhao et al., 2013); and (vi) various applications to control and signal processing *etc.* (Wang et al., 2012; Xu & Hong, 2013), just to quote a few.

In a very recent work, the STP has also been used to the modeling, analysis and control design of the NEGs (Cheng et al., Preprint2013). This paper is a development of Cheng et al. (Preprint2013). It provided a comprehensive discussion for various NEGs. The NEGs discussed could have three different graphs (i) undirected graph, which is used for the NEGs with symmetric fundamental network games (FNG); (ii) directed graph, which is used for the NEGs with asymmetric FNGs; and (iii) d-directed graph, which is used for symmetric games with partial neighborhood information. Three kinds of FNGs are discussed, which are (i) each player has two strategies and the game is symmetric (S-2); (ii) each player has two strategies and the game is asymmetric (A-2); and (iii) each player has three strategies and the game is symmetric (S-3). Three strategy updating rules (SUR) are introduced, which are (i) Unconditional Imitation (UI); (ii) Fermi Rule (FR); (iii) Myopic Best Response Adjustment Rule (MBRA). Though most of widely discussed kinds of NEGs will be discussed in detail, the technique developed is applicable for other cases.

⋆ This work was supported in part by National Natural Science Foundation (NNSF) of China under Grants 61074114, 61273013, 61104065, 61374168, and 61333001.

Then we review the fundamental evolutionary equation (FEE) introduced in Cheng et al. (Preprint2013) and construct the FEEs for various types of NEGs.

For statement ease, some notations and basic concepts are introduced first.

- Notations:
  (i) $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
  (ii) $\text{Col}(M)$ ($\text{Row}(M)$) is the set of columns (rows) of M. $\text{Col}_i(M)$ ($\text{Row}_i(M)$) is the i-th column (row) of M.
  (ii) $\mathcal{D}_k := \{1, 2, \cdots, k\}, \quad k \geq 2$.
  (iii) δ_n^i: the i-th column of the identity matrix I_n.
  (iv) $\Delta_n := \{\delta_n^i | i = 1, \cdots, n\}$.
  (v) $\Upsilon_k = \left\{(r_1, \cdots, r_k) \big| r_i \geq 0,\ i = 1, \cdots, k;\ \sum_{i=1}^k r_i = 1\right\}$ is called the set of k-th dimensional probabilistic vectors.
  (vi) A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if the set of columns of L, denoted by $\text{Col}(L)$, are of the form of δ_m^k. That is,
  $$\text{Col}(L) \subset \Delta_m.$$
  Denote by $\mathcal{L}_{m\times n}$ the set of $m\times n$ logical matrices.
  (vii) If $L \in \mathcal{L}_{n\times r}$, by definition it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \cdots, \delta_n^{i_r}]$. For the sake of brevity, it is briefly denoted as
  $$L = \delta_n[i_1, i_2, \cdots, i_r].$$
  (viii) A matrix $L \in \mathcal{M}_{m\times n}$ is called a probabilistic matrix if the columns of L are m-dimensional probabilistic vectors. That is,
  $$\text{Col}(L) \subset \Upsilon_m.$$
  The set of $m\times n$ probabilistic matrices is denoted by $\Upsilon_{m\times n}$.
  (ix) If $L \in \Upsilon_{m\times n}$, if $\text{Col}(L) = C_1 \cup C_2$, where $C_1 \subset \Delta_m$ and $C_2 \subset \Upsilon_m \backslash \Delta_m$, and $|C_2| \ll |C_1|$. Then for notational compactness, we still use the shorthand
  $$L = \delta_m[i_1, i_2, \cdots, i_n],$$
  where if $\text{Col}_k(L) = \delta_m^s \in C_1$, $i_k = s$, else if $\text{Col}_k(L) \in C_2$, that is, $\text{Col}_k(L) = (r_1, \cdots, r_m)$, we express i_k as
  $$i_k = 1/(r_1) + 2/(r_2) + \cdots + m/(r_m).$$
- Operators:
  (i) Semi-tensor product of matrices (Cheng et al., 2011, 2012a):
  Definition 1. Let $M \in \mathcal{M}_{m\times n}$ and $N \in \mathcal{M}_{p\times q}$, and $t = \text{lcm}\{n, p\}$ be the least common multiple of n and p. The semi-tensor product of M and N, denoted by $M \ltimes N$, is defined as
  $$\left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p}, \qquad (1)$$
  where $\otimes$ is the Kronecker product.
  (ii) Khatri-Rao Product of matrices (Ljung & Söderström, 1982)
  Definition 2. Let $M \in \mathcal{M}_{p\times m}$, $N \in \mathcal{M}_{q\times m}$. Then the Khatri-Rao Product is defined as
  $$M * N = [\,\text{Col}_1(M) \ltimes \text{Col}_1(N) \cdots \text{Col}_m(M) \ltimes \text{Col}_m(N)] \in \mathcal{M}_{pq\times m}.$$
  Proposition 3. Let $X \in \mathbb{R}^m$ be a column and M is a matrix. Then
  $$X \ltimes M = (I_m \otimes M)\, X. \qquad (2)$$
  (iii) Swap matrix (Cheng et al., 2011, 2012a):
  Definition 4. A matrix $W_{[m,n]} \in \mathcal{M}_{m\times n}$, defined by
  $$W_{[m,n]} = \delta_{mn}[1, m+1, \cdots, (n-1)m+1;\ 2, m+2, \cdots, (n-1)m+2;\ \cdots;\ n, m+n, \cdots, nm], \qquad (3)$$
  is called the (m, n)-dimensional swap matrix.
  The basic function of the swap matrix is to "swap" two vectors. That is,
  Proposition 5. Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then
  $$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X. \qquad (4)$$

The rest of this paper is organized as follows: Section 2 presents a mathematical framework for NEGs. Three basic components of an NEG, namely, network graph, FNG, and SUR, are discussed in detail. Section 3 is devoted to the FEE, which plays a key role in the investigation of NEGs. FEEs of all players are building block for constructing strategy profile dynamics of the overall networks. Section 4 is a brief conclusion.

2. MATHEMATICAL FRAMEWORK FOR NEG

This section is a comprehensive description of mathematical framework of NEGs. The main idea of which was firstly proposed in Cheng et al. (Preprint2013).

Definition 6. A networked evolutionary game, denoted by $((N, E), G, \Pi)$, consists of three ingredients as:

(i) a network (graph) (N, E);
(ii) a fundamental network game (FNG), G, such that if $(i, j) \in E$, then i and j play the FNG with strategies $x_i(t)$ and $x_j(t)$ respectively.
(iii) a local information based strategy updating rule (SUR).

In the following we describe these three ingredients one by one.

2.1 Network Graph

We consider three kinds of network graphs.

(i) Undirected graph: $N = \{1, 2, \cdots, n\}$ $(n \leq \infty)$. It represents n players. If $(i, j) \in E$, then i is in the neighborhood of j, denoted by $i \in U(j)$. Simultaneously, $j \in U(i)$.
(ii) Directed graph: Note that the FNG is always played by two neighboring players. If the FNG is not symmetric, the directed edge is used to distinguish different roles of two players. Assume $(i, j) \in E$, *i.e.*, there is an edge from i to j, then in the game i is player 1 and j is player 2. Note that such directed graph does not affect the definition of neighborhoods.
(iii) D-directed graph: Assume the FNG is still symmetric, but the graph is not symmetric. That is, if $(i, j) \in E$, denoted by dot line arrow goes from i to j, it means