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Bounds for Input- and State-to-Output Properties of Uncertain Linear Systems ⋆

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Abstract We consider the effect of parametric uncertainty on properties of Linear Time Invariant systems. Traditional approaches to this problem determine the worst-case gains of the system over the uncertainty set. Whilst such approaches are computationally tractable, the upper bound obtained is not necessarily informative in terms of assessing the influence of the parameters on the system performance. We present theoretical results that lead to simple, convex algorithms producing parametric bounds on the $\mathcal{L}_2$-induced input-to-output and state-to-output gains as a function of the uncertain parameters. These bounds provide quantitative information about how the uncertainty affects the system.

Keywords: Robust control, Uncertainty analysis, Linear systems, Semi-definite Programming.

1. INTRODUCTION

The application of polynomial optimisation techniques to robust control problems have been successful both in terms of sum-of-squares programs (Blekherman et al. (2013)) and the generalized problem of moments (Lasserre (2009)). Some of these advances were highlighted in the survey paper on Robust Control (Petersen and Tempo, 2014, Section 6.1). The stability analysis of linear systems with linear dependence on uncertain parameters lying on the simplex was performed using parameter-dependent Lyapunov functions (LFs) as detailed in Chesi (2010) (see Section VI.C). In Chesi et al. (2005) it is proven that, for time-invariant systems, LFs which are quadratic in the state and homogeneous on parameters are necessary and sufficient to prove stability of the linear system. However the resulting test in terms of homogeneous polynomials may require a LF of large degree. Polynomial LFs have also been studied by Bliman (2004) where, exploiting the fact that the parameterised Lyapunov matrix is analytic on the uncertainty domain, it is demonstrated that there exists a homogenous polynomial LF proving the stability for uncertainties in the hypercube. In Oliveira and Peres (2007), Pólya's theorem is applied to solve homogeneous inequalities on the simplex. Conditions for discrete-time linear systems with polynomial parameter-dependent LFs were presented in Lavaei and Aghdam (2008) where the simplex as the uncertain domain is presented as a particular case.

Dissipation inequalities for uncertain linear systems were studied considering interval matrix uncertainties in Barb et al. (2003), where the *limits* on the uncertainty for which a dissipation inequality holds were investigated. In Sato and Peaucelle (2006), parameter-dependent LF were considered to compute bounds on $\mathcal{H}_\infty$ and $\mathcal{H}_2$ gains over polytopic uncertainty domains for linear systems with rational dependence on the uncertain parameters. The authors then use the slack variable method to solve the resulting set of matrix inequalities which are affine on the uncertain parameters. The same conditions for worst-case computation of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms were also presented in (Chesi, 2010, Section VI.D). In the case of autonomous systems, $\mathcal{L}_\infty$ norm bounds on nonlinear system trajectories, valid over compact sets of initial conditions, have been constructed using LF based approaches in Chesi and Hung (2008).

A parameterised property (or parameterised bounds on a property), such as the $\mathcal{H}_\infty$-norm of an uncertain system gives the designer important information for the selection of the components of a physical system (which define the system parameters) and allows them to quantify beforehand the degradation, measured as the norm of the mismatch to the target (nominal) performance. Predefined bounds on performance degradation can be useful for robust and resilient control law design. In this context, obtaining a closed-loop performance curve that respects a specified degradation bound becomes a requirement for control synthesis. This is in contrast to the customary approach of searching for guaranteed cost solutions which accounts for the worst case scenario given a set of uncertain parameters. Also, in the context of gain-scheduling (Packard (1994)), having a parameterised performance index for a set of given controllers may be of use when designing the scheduling parameter.

A system is said to be robust to variations in its parameters if it is able to maintain its performance properties subject to a bounded variation of these parameters. For instance,

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the difference between the output signal of the nominal model and that of any perturbed model is expected to be *small* in some sense. From the viewpoint of system identification, such a property can be problematic: a low degree of uncertainty in the input lends itself to a high degree of uncertainty in the parameter estimate, and the identification protocol becomes ill-conditioned. This property has been variously referred to as quantitative unidentifiability (Vajda et al. (1989)), practical unidentifiability (Holmberg and Ranta (1982)), and sloppiness (Brown and Sethna (2003)).

Traditional methods of diagnosis (Vajda et al. (1989)) involve linearisation of the effects of parameter perturbation around a previously provided parameter estimate. However such effects are typically highly nonlinear (Hines et al. (2014)). Recent advances from various angles have been made in the construction of nonlinear approximations to regions of practically unidentifiable parameters around a nominal estimate (e.g. Raue et al. (2009); Calderhead and Girolami (2011)). Such advances typically rely on educated sampling of outputs over parameter space, with each sample involving the simulation of an ODE. Parameterised performance indices as provided in this paper, in contrast, provide algebraic characterisations of performance degradation in parameter space. This allows for simple construction of level sets of parameters inducing a given magnitude output perturbation, or optimisation of some system property subject to parametric constraints specifying a given maximal level of performance degradation.

This paper is organized as follows: Section 2 formulates the problem. In Section 3 we present results allowing to characterise input/state-to-output properties as functions of uncertain parameters. Section 4 illustrates the results with an analytical and numerical examples. The paper is concluded in Section 5.

Notation. We take $\mathbb{R}^n$ to denote the n-dimensional Euclidean space. $\mathbb{R}_{\geq 0}$ specifies the set of non-negative real numbers. I denotes the identity matrix. For a given matrix $M \in \mathbb{R}^{n\times n}$, $Tr(M)$ is the trace of M, $det(M)$ is the determinant, and $He(M) = M + M^T$. For a symmetric matrix, if $M \geq (\leq) 0$, then M is positive (negative) semidefinite. For $x_0 \in \mathbb{R}^n$ and $x(t) \in \mathcal{L}_2$, $\|x_0\|$, $\|x\|_2^2$ respectively denote Euclidean norm of x_0 and the squared $\mathcal{L}_2$-norm of $x(t)$. $\mathcal{C}^k$ is defined to be the set of continuous, k-times differentiable, scalar functions.

2. PROBLEM FORMULATION

Let $\Theta \subset \mathbb{R}^{n_\theta}$ define the set of uncertain parameters. Let $\theta^* \in \Theta$ be the fixed *nominal parameter values.* We refer to the linear system

$$\begin{cases} \dot{x} = A(\theta^*)x + B(\theta^*)u \\ y = C(\theta^*)x + D(\theta^*)u \end{cases} \tag{1}$$

with $x(0) = x_0$, $A : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n\times n}$, $B : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n\times m}$, $C : \mathbb{R}^{n_\theta} \to \mathbb{R}^{p\times n}$ as the *nominal system.* Consider the linear time-invariant parameter-dependent system

$$\begin{cases} \dot{\tilde{x}} = A(\theta)\tilde{x} + B(\theta)u \\ \tilde{y} = C(\theta)\tilde{x} + D(\theta)u, \end{cases} \tag{2}$$

$\tilde{x}(0) = x(0) = x_0$ (we consider the initial conditions of (1) and (2) to match for reasons that will become clear in next section). System (2) is called the *uncertain system.*

Define the signals $e(t) := x(t) - \tilde{x}(t)$ and $\Delta y(t) := y(t) - \tilde{y}(t)$ to obtain

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A(\theta^*) & 0 \\ \Delta A(\theta) & A(\theta) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B(\theta^*) \\ \Delta B(\theta) \end{bmatrix} u \\ \begin{bmatrix} y \\ \Delta y \end{bmatrix} = \begin{bmatrix} C(\theta^*) & 0 \\ \Delta C(\theta) & C(\theta) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} D(\theta^*) \\ \Delta D(\theta) \end{bmatrix} u. \end{cases} \tag{3}$$

where $\Delta A(\theta) := A(\theta^*) - A(\theta)$, $\Delta B(\theta) := B(\theta^*) - B(\theta)$ and $\Delta C(\theta) := C(\theta^*) - C(\theta)$, $\Delta D(\theta) := D(\theta^*) - D(\theta)$, $x(0) = x_0$ and $e(0) = 0$.

We assume throughout that $A(\theta)$ is Hurwitz $\forall \theta \in \Theta$. From the linearity of (3) and the triangular structure of the system matrix we have stability of the origin.

In the following section we describe input and state-to-output properties of (3) considering the output Δy, which is the deviation from the nominal behavior.

The output of the uncertain system (2) is written in terms of (x, e) as

$$\tilde{y} = \left[\, C(\theta) \ \ -C(\theta) \,\right] \begin{bmatrix} x \\ e \end{bmatrix} + D(\theta)u.$$

The interest in introducing $\tilde{y}$ as above, instead of the mismatched output Δy as in (3), is motivated by the fact that we may want a bound for $\|y(t)\|_2^2 - \|\tilde{y}(t)\|_2^2$ instead of $\|y(t) - \tilde{y}(t)\|_2^2 = \|\Delta y(t)\|_2^2$ as developed in this paper.

2.1 Invariance of the output

In order to make a distinction between the transient effects and the steady-state effects of the uncertainty on the mismatch signal $\Delta y(t)$ let us define the set of parameters that leave the steady-state output invariant with constant inputs.

Definition 1. (Steady-State Output Invariant Set).The set in the space of parameters

$$\Theta_{0ss}^{(i)} := \left\{ \theta \in \mathbb{R}^{n_\theta} \mid \lim_{t\to\infty} \Delta y(t) = 0 \, \forall u_i \in \mathbb{R} \right\} \tag{4}$$

is called the *steady-state output invariant set of parameters with respect to* i. Here, $u_i \in \mathbb{R}^m$ is a constant input vector taking the value 0 in all components except the i^{th}. The set

$$\Theta_{0ss} := \bigcap_{i=1}^{m} \Theta_{0ss}^{(i)} \tag{5}$$

is then defined as the *steady-state output invariant parameter set.*

Let us now define the set of parameters leaving the output invariant with time-varying inputs.

Definition 2. (Output invariant parameter set). The set

$$\Theta_0^{(i)} := \{ \theta \in \mathbb{R}^{n_\theta} \mid \Delta y(t) = 0 \, \forall t, u_i(t) \} \tag{6}$$

is called the *invariant set of parameters with respect to input* i. Here, u_i is as defined in Definition 1.

$$\Theta_0 := \bigcap_{i=1}^{m} \Theta_0^{(i)} \tag{7}$$

is the *invariant parameter set.*

Proposition 1. For system (3) the steady-state output invariant set of parameters as in Definition 1 with respect to input i can equivalently be characterised by

$$\Theta_{0ss}^{(i)} := \{ \theta \in \mathbb{R}^{n_\theta} \mid f_{ss}(\theta) = 0 \} \tag{8}$$