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IFAC-PapersOnLine 48-14 (2015) 007–012

The homological perturbation lemma and its applications to robust stabilization

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Abstract Within the lattice approach to analysis and synthesis problems, we show how standard results on robust stabilization can be obtained in a unified way and generalized when interpreted as a particular case of the so-called *homological perturbation lemma*. This lemma plays a significant role in algebraic topology, homological algebra, computer algebra, etc. Our results show that it is also central to robust control theory for (infinite-dimensional) linear systems.

Keywords: Robust control, H_∞ control, robust stabilization, homological perturbation lemma, models of perturbations, the lattice approach to analysis and synthesis problems.

1. THE FRACTIONAL REPRESENTATION APPROACH

In what follows, we consider the so-called *fractional representation approach* developed in the 80's by Vidyasagar, Desoer, Callier, Francis, etc (see Curtain et al. (1991); Desoer et al. (1980); Vidyasagar (1985) and the references therein). In this approach, the set of stable plants (in a sense to be defined afterwards) is considered to be an *integral domain* A, i.e., a commutative ring with no non-zero zero divisors. Examples of integral domains A usually encountered in the literature are:

- The *Hardy algebra* $H^\infty(\mathbb{C}_+)$ formed by all holomorphic functions in the *open right half-plane*
$$\mathbb{C}_+ := \{s \in \mathbb{C} \mid \Re(s) > 0\}$$
which are bounded with respect to the *sup norm*, i.e.:
$$\| h \|_\infty := \sup_{s \in \mathbb{C}_+} |h(s)|.$$
Let $\widehat{h}$ denote the Laplace transform of h and $H^2(\mathbb{C}_+) := \{\widehat{h} \mid h \in L^2(\mathbb{R}_+)\}$. If $\widehat{h} \in H^\infty(\mathbb{C}_+)$ then the input-output system $\widehat{y} = \widehat{h}\,\widehat{u}$ is $H^2(\mathbb{C}_+) - H^2(\mathbb{C}_+)$ stable (i.e., $\widehat{u} \in H^2(\mathbb{C}_+)$ yields $\widehat{y} \in H^2(\mathbb{C}_+)$), or equivalently $y = h \star u$ is the $L^2(\mathbb{R}_+) - L^2(\mathbb{R}_+)$ stable, where $\star$ denotes the standard convolution product.
- $RH_\infty := H^\infty(\mathbb{C}_+) \cap \mathbb{R}(s)$ the algebra of proper and stable rational transfer functions.
- The Wiener algebra defined by:
$$\widehat{\mathcal{A}} := \{\widehat{f} + \sum_{i=0}^{+\infty} a_i\, e^{-h_i s} \mid f \in L_1(\mathbb{R}_+),\ (a_i) \in l_1(\mathbb{N}),\ 0 = h_0 < h_1 < h_2 < \ldots\}.$$
If $\widehat{h} \in \widehat{\mathcal{A}}$, then the input-output system $y = h \star u$ is $L^\infty(\mathbb{R}_+) - L^\infty(\mathbb{R}_+)$ stable (BIBO stability).

* The author would like to thank Julio Rubio (University of Rioja, Spain) and Francis Sergeraert (University of Grenoble, France) for introducing him to the field of the homological perturbation lemma and for motivating discussions which led to this paper.

For more examples, see Curtain et al. (1991); Desoer et al. (1980); Quadrat (2006a); Vidyasagar (1985).

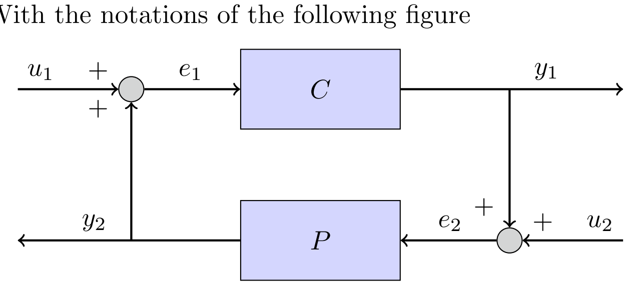
In what follows, $K := Q(A) = \left\{\frac{n}{d} \mid 0 \neq d,\, n \in A\right\}$ denotes the *quotient field* of A. Within the fractional representation approach, a transfer matrix is defined by $P \in K^{q \times r}$.

Definition 1. (Desoer et al. (1980); Vidyasagar (1985)). Let A be an integral domain of SISO stable transfer functions, $K := Q(A)$ and $P \in K^{q \times r}$. Then, P is *internally stabilizable* if there exists $C \in K^{r \times q}$ such that all entries of the following transfer matrix

$$H(P, C) := \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} = \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \quad (1)$$

belong to A, i.e., $H(P, C) \in A^{(q+r)\times(q+r)}$. Then, C is a *stabilizing controller* of P and we note $C \in \mathrm{Stab}(P)$.

With the notations of the following figure



we have:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = H(P, C) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The transfer matrix $H(P, C)$ connects the inputs u_1 and u_2 (references and perturbations) to e_1 and e_2. If we have $H(P, C) \in A^{(q+r)\times(q+r)}$, then all transfer matrices

10.1016/j.ifacol.2015.09.425

between two signals appearing in the above figure are stable. For more details, see, e.g., Desoer et al. (1980); Vidyasagar (1985). Since the context is clear, we shall only say "stabilizable" for "internally stabilizable".

Let us introduce standard transfer matrices:

- *Output sensitivity* transfer matrix $S_o := (I_q - P\,C)^{-1}$.
- *Input sensitivity* transfer matrix $S_i := (I_r - C\,P)^{-1}$.
- $U := C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C$.
- *Complementary input sensitivity* transfer matrix $T_i := U\,P$.
- *Complementary output sensitivity* transfer matrix $T_o := P\,U$.

Note that we have the relation $S_o\,P = P\,S_i$.

Let us introduce a few more definitions.

Definition 2. (Desoer et al. (1980)). Let $P \in K^{q\times r}$.

(1) A *fractional representation* of P is defined by
$$P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1},$$
where $R := (D \quad -N) \in A^{q\times(q+r)}$, $\det D \neq 0$, $\widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T \in A^{(q+r)\times r}$ and $\det \widetilde{D} \neq 0$.

(2) A fractional representation $P = D^{-1}\,N$ is a *left coprime factorization* if there exist $X \in A^{q\times q}$ and $Y \in A^{r\times q}$ such that $D\,X - N\,Y = I_q$.

(3) A fractional representation $P = \widetilde{N}\,\widetilde{D}^{-1}$ is a *right coprime factorization* if there exist $\widetilde{X} \in A^{r\times r}$ and $\widetilde{Y} \in A^{r\times q}$ such that $-\widetilde{Y}\,\widetilde{N} + \widetilde{X}\,\widetilde{D} = I_r$.

(4) A fractional representation of $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$ is a *doubly coprime factorization* if $P = D^{-1}\,N$ is a left coprime factorization and $P = \widetilde{N}\,\widetilde{D}^{-1}$ is a right coprime factorization.

Remark 1. Any transfer matrix $P \in K^{q\times r}$ admits fractional representations (take, e.g., $D = d\,I_q$, $\widetilde{D} = d\,I_r$, where d is the product of the denominators of all the entries of P and $N := d\,P$ and $\widetilde{N} = P\,d$). But not all transfer matrices $P \in K^{q\times r}$ admit a left/right/doubly coprime factorization. For instance, see Quadrat (2006a,b).

2. THE LATTICE APPROACH

As we showed in Quadrat (2006a,b), the fractional representation approach to analysis and synthesis problems can be studied using the concept of the *lattice of a finite-dimensional K-vector space*. Before stating this definition again, let us introduce a few standard definitions.

Definition 3. (Rotman (2009)). Let A be an integral domain, $K := Q(A)$ and M a finitely generated A-module.

(1) The *rank* of M is the dimension of the K-vector space obtained by extending the coefficients of M from A to K, i.e., $\mathrm{rank}_A(M) := \dim_K(K \otimes_A M)$, where $\otimes_A$ denotes the *tensor product* of A-modules.

(2) If M and N are two A-modules, then $\hom_A(M, N)$ denotes the set of all the A-*homomorphisms* from M to N, i.e., $f \in \hom_A(M, N)$ satisfies
$$f(a_1\,m_1 + a_2\,m_2) = a_1\,f(m_1) + a_2\,f(m_2)$$
for all $a_1, a_2 \in A$ and for all $m_1, m_2 \in M$.

(3) M is *free* if M admits a basis or equivalently if M is isomorphic to direct sum of copies of A, i.e., $M \cong A^r$, where $\cong$ stands for an *isomorphism*, i.e., a homomorphism which is both injective and surjective.

(4) M is *projective* if there exist an A-module P and $r \in \mathbb{N}$ such that $M \oplus P \cong A^r$, where $\oplus$ denotes the *direct sum* of A-modules.

Definition 4. (Bourbaki (1989)). Let V be a finite-dimensional K-vector space. Then, an A-submodule M of V is a *lattice of* V if there exist two free A-submodules L_1 and L_2 of V such that $L_1 \subseteq M \subseteq L_2$ and $\mathrm{rank}_A(L_1) = \dim_K(V)$.

We have the following examples (Quadrat (2006a)).

Example 1. If $P \in K^{q\times r}$, then we have:

- $\mathcal{L} := (I_q \quad -P)\,A^{q+r}$ is a lattice of K^q.
- $\mathcal{M} := A^{1\times(q+r)}\,\left(P^T \quad I_r^T\right)^T$ is a lattice of $K^{1\times r}$.

Definition 5. (Bourbaki (1989)). Let V be a finite-dimensional K-vector space and M a lattice of V. Then, $A : M$ is the A-submodule of $\hom_K(V, K) \cong V^\star$ formed by the K-linear maps $f : V \longrightarrow K$ which satisfy $f(M) \subseteq A$.

One can show that $A : M$ is a lattice of $\hom_K(V, K) \cong V^\star$.

We have the following examples (Quadrat (2006a)).

Example 2. With the notations of Example 1, we have:

- $A : \mathcal{L} = \{\lambda \in A^{1\times q} \mid \lambda\,P \in A^{1\times r}\}$ is a lattice of $K^{1\times q}$.
- $A : \mathcal{M} = \{\mu \in A^r \mid P\,\mu \in A^q\}$ is a lattice of K^r.

The next theorems give necessary and sufficient stabilization conditions (Quadrat (2006a)).

Theorem 1. With the notations of Example 1, the following assertions are equivalent:

(1) $P \in K^{q\times r}$ is stabilizable.

(2) There exists $L = (S_o^T \quad U^T)^T$, where $S_o \in A^{q\times q}$ and $U \in A^{r\times q}$, such that:
  (a) $L\,P = \begin{pmatrix} S_o\,P \\ U\,P \end{pmatrix} \in A^{(q+r)\times r}$.
  (b) $(I_q \quad -P)\,L = S_o - P\,U = I_q$.
  Then, we have $C := U\,S_o^{-1} \in \mathrm{Stab}(P)$ and:
$$S_o = (I_q - P\,C)^{-1}, \quad U = C\,(I_q - P\,C)^{-1}.$$

(3) $\mathcal{L}$ is a *projective lattice* of K^q, i.e., the lattice $\mathcal{L}$ of K^q is a finitely generated projective A-module of rank q.

Theorem 2. With the notations of Example 1, the following assertions are equivalent:

(1) $P \in K^{q\times r}$ is stabilizable.

(2) There exists $\widetilde{L} = (-U \quad S_i)$, where $U \in A^{r\times q}$ and $S_i \in A^{r\times r}$, such that:
  (a) $P\,\widetilde{L} = (-P\,U \quad P\,S_i) \in A^{q\times(q+r)}$.
  (b) $\widetilde{L}\,\begin{pmatrix} P \\ I_r \end{pmatrix} = -U\,P + S_i = I_r$.
  Then, we have $C := S_i^{-1}\,U \in \mathrm{Stab}(P)$ and:
$$S_i = (I_r - C\,P)^{-1}, \quad U = (I_r - C\,P)^{-1}\,C.$$

(3) $\mathcal{M}$ is a projective lattice of $K^{1\times r}$.

We have the following result (Quadrat (2006a)).

Corollary 3. (1) If $P = D^{-1}\,N$ is a left coprime factorization, $D\,X - N\,Y = I_q$, $X \in A^{q\times q}$, $Y \in A^{r\times q}$, then $\mathcal{L} = D^{-1}\,A^q \cong A^q$ is free and Theorem 1 holds with:
$$S_o = X\,D, \quad U = Y\,D.$$
Hence, P is stabilized by the controller $C := Y\,X^{-1}$.