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Necessary Conditions for Robust Stability of Linear Systems ⋆

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Abstract: In this paper we consider parameterized Lyapunov inequalities arising in robust stability analysis of linear systems with structured uncertainty. A necessary condition of their feasibility is presented. This condition is based on an optimization method developed by the author. Its properties include being reasonably easy to check, and its coverage being adjustable via the algorithm's settings.

Keywords: robust stability, Lyapunov method, matrix inequalities, polynomial inequalities, nonlinear programming, global optimization.

1. INTRODUCTION

A large number of control theory problems can be formalized as optimization problems with polynomial matrix inequality (PMI) constraints. Their general structure is as follows:

$$\begin{aligned} & f^* = \min_x f(x), \\ & G_i(x) \geq 0, \\ & x \in \mathbb{R}^n, \quad G_i(x) = G_i^{\mathrm{T}}(x) \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, \ldots, m, \end{aligned} \tag{1}$$

where $f(x)$ and elements of $G_i(x)$ are (not necessarily convex) polynomial functions, and the inequality sign denotes positive semidefiniteness. In practice, they are often written down in a more compact form with unknown variables being represented by matrices. Additional equality constraints may also be added depending on a specific problem. If only problem feasibility is relevant, the goal function $f(x)$ may be omitted.

The most well-researched kind of such problems are linear matrix inequality (LMI) systems. The simplest widely known problem is verifying stability of a continuous linear dynamic system by solving a Lyapunov inequality:

$$\begin{aligned} & A^{\mathrm{T}}P + PA < 0, \\ & P > 0, \\ & P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}, \end{aligned}$$

with P being an unknown matrix. While this basic form is fairly simple, it can quickly escalate into bilinear matrix inequalities or the general polynomial form. For example, synthesizing a stabilizing static output controller is a bilinear problem:

$$\begin{aligned} & (A + BKC)^{\mathrm{T}}P + P(A + BKC) < 0, \\ & P > 0, \\ & P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}, \; K \in \mathbb{R}^{l \times m}, \end{aligned} \tag{2}$$

where P and K are unknown matrices; A, B, and C are appropriately sized given matrices.

Another related type of problem arises in analysis of systems with structured uncertainties. Consider the following model:

$$\begin{aligned} \dot{x}(t) &= A(r(t))x(t) + B(r(t))u(t), \\ y(t) &= C(r(t))x(t) + D(r(t))u(t), \end{aligned} \tag{3}$$

where x is the state, u is the input, y is the output, $r(t) \in \mathcal{U}$ is the uncertainty, $\mathcal{U} = \{r \mid U(r) \geq 0\}$ with U being a symmetric matrix polynomial; A, B, C, and D are appropriately sized matrix polynomials.

The most basic problem related to such systems is determining their stability. Relevant results are similar to the ones for non-uncertain systems with an added layer of complexity. In particular, it has been shown (Chesi et al. (2003, 2009)) that (the open loop) system (3) with time-invariant simplex uncertainty is asymptotically stable if and only if the problem

$$\begin{aligned} & \forall r \in \mathcal{U}: \\ & A(r)^{\mathrm{T}}P(r) + P(r)A(r) < 0, \\ & P(r) > 0, \end{aligned} \tag{4}$$

has a solution in the form of a homogeneous matrix polynomial $P(r) = P(r)^{\mathrm{T}}$ with a degree no greater than a known value. This problem can be reduced to checking positive definiteness of certain matrix polynomials, and then to a number of linear matrix inequality (LMI) problems. Verification of polynomial positive definiteness for all argument values is associated with sum-of-squares (SOS) problems. This class is dual to PMI problems; later in this paper we demonstrate transformation of (4) to the PMI form and its solution. For time-varying uncertainties, techniques based on polynomial (non-quadratic) Lyapunov functions have been proposed; see, e.g., Zelentsovsky (1994).

Other problems like finding $\mathcal{H}_2$ or $\mathcal{H}_\infty$ system performance also have their representations for uncertain systems. For example, an upper bound of (3)'s $\mathcal{H}_\infty$ performance for time-invariant uncertainty can be established as

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$$\begin{aligned}&\hat{\gamma}_\infty = \inf \gamma,\\ &\forall r \in \mathcal{U}:\\ &P(r) > 0,\ Q(r) > 0,\end{aligned} \tag{5}$$

with $P(r)$ being a matrix polynomial, and

$$\begin{aligned}Q(r) = &-\begin{bmatrix} A(r)^\mathrm{T} P(r) + P(r)A(r) & P(r)B(r) \\ B(r)^\mathrm{T} P(r) & -I \end{bmatrix}\\ &-\frac{1}{\gamma^2}\begin{bmatrix} C(r)^\mathrm{T} C(r) & C(r)^\mathrm{T} D(r) \\ D(r)^\mathrm{T} C(r) & D(r)^\mathrm{T} D(r) \end{bmatrix}.\end{aligned}$$

All these tests can be converted to certain kinds of LMI problems of varying conservativeness. A common theme here, however, is that the size of these problems tends to grow fast depending on sizes and degrees of matrix polynomials in (3), as well as degree of $P(r)$ and the kind of LMI approximation used. (The latter two characteristics also directly influence conservatism of the solution.) Combinatorial explosion of problem sizes is not uncommon in this context.

In the current paper, we consider an approach to such problems based on a solution method for problems involving PMIs or non-polynomial forms. This approach can be used to directly solve problems similar to (2), or construct feasibility tests for forms like (4) and (5) without explicitly constructing their LMI approximations. While the optimization method used here is not global in the general case, and said feasibility tests are not necessary and sufficient, they are fairly easy to use, and the algorithm can be tuned for various levels of search space coverage.

Section 2 provides additional background information on some existing solution techniques for these problems. Information on the new optimization method is provided in section 3. Section 4 describes application of this method to stability problems; section 5 shows a numerical example. Additional types of more complex potential applications are listed in section 6.

2. BASE METHODS

There exist a number of techniques applicable to various kinds of the above problems. In this section, we briefly describe two of them to provide the general context of existing approaches; for a more detailed review, see, e.g., Chesi (2010).

The main kind of problems that will play a role in this paper, are PMI optimization problems (1). The solution method most important here is described in Lasserre (2001); Henrion and Lasserre (2005, 2006). This method is based on constructing LMI relaxations, i.e., LMI problems

$$\begin{aligned}&f^* = \min_y \sum_i f_i y_i,\\ &M_k(y) \geq 0,\\ &M_{k-d_i}(G_i, y) \geq 0, \quad i = 1, \ldots, m,\\ &y_1 = 1,\end{aligned}$$

where k is the relaxation order; $d_i = \lceil \frac{1}{2} \deg G_i(x) \rceil$; $y = [y_i]_i = \int b_{2k}(x)\,\mathrm{d}\,\mu$ is the vector of moments of some unknown measure μ; $b_r(x)$ is the monomial basis of the space of polynomials having degrees up to r: $b_r(x) = [1\ x_1\ x_2\ \ldots\ x_n\ x_1^2\ x_1x_2\ \ldots\ x_n^2\ \ldots\ x_1^r\ \ldots\ x_n^r]^\mathrm{T}$; vector $[f_i]_i$ is a representation of $f(x)$ in this basis: $f(x) \equiv \sum_i (b_k(x))_i f_i$. $M_k(y)$ and $M_{k-d_i}(G_i, y)$ are the moment matrix and localizing matrices derived from

$$\begin{aligned}M_k(y) &\equiv \int b_k(x) b_k(x)^\mathrm{T}\ \mathrm{d}\,\mu,\\ M_{k-d}(G, y) &\equiv \int \left(b_{k-d}(x) b_{k-d}(x)^\mathrm{T}\right) \otimes G(x)\ \mathrm{d}\,\mu.\end{aligned}$$

For $k \to \infty$, minimum of the LMI relaxation approaches the minimum of the original PMI problem; in practice, for many problems they become equal for finite, relatively small, values of k, and the vector of moments of PMI solution becomes a solution to the LMI relaxation.

This global optimization method is quite powerful. However, it suffers from combinatorial explosion of LMI relaxation sizes (that can be traced back to the size of $b_{2k}(x)$). The authors provide an example (Henrion and Lasserre (2006), section III.E, example AC8 taken from Leibfritz (2004)) of a static output feedback problem with 9 states, 1 input and 5 outputs, where a naive attempt at LMI relaxation construction would result in a problem with 33649 unknown variables in the very first iteration.

As an example of another type of problem, consider the system

$$\begin{aligned}&\forall x \in \mathcal{U}:\\ &P(x) > 0,\end{aligned}$$

where $P(x) = P(x)^\mathrm{T} \in \mathbb{R}^{r \times r}$ is a known matrix polynomial of degree d, and the set $\mathcal{U}$ is a simplex $\mathcal{U} = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1, x_i \geq 0\}$. As shown in Chesi et al. (2003), a matrix polynomial $P(x) = \sum_{i=0}^d H_i(x)$, where $H_i(x)$ is a homogeneous matrix polynomial of degree i, is positive definite for all $x \in \mathcal{U}$ if and only if

$$\begin{aligned}&\forall x \in \mathbb{R}^n \backslash \{0\} : \hat{H}(x) > 0,\\ &\hat{H}(x) = H(y)|_{y=(x_1^2,\ldots,x_n^2)^\mathrm{T}},\\ &H(y) = \sum_{i=0}^d H_i(y) \left(\sum_{i=1}^n y_i\right)^{d-i}.\end{aligned} \tag{6}$$

This way the problem is converted to a similar one with variable x no longer constrained by the simplex. A sufficient condition can be formulated that requires $\hat{H}(x)$ to be an SOS matrix polynomial. This check can be performed by constructing an LMI relaxation for the problem. While all steps are well defined, the procedure does have certain issues—in particular, the same combinatorial explosion of LMI sizes; for instance, the number of free variables in the relaxation is $\frac{1}{2} r (C_{n+d}^n (r C_{n+d}^n + 1) - (r+1) C_{n+2d}^n)$.

3. ATOMIC OPTIMIZATION

For the method of solving PMI problems (1) described in the previous section, a transformation has been proposed in Pozdyayev (2013, 2014) that was aimed at significantly reducing its computational complexity while maintaining its key benefits for a class of problems related to control theory, in particular, to the Lyapunov method.

The new optimization method has been named "atomic optimization" due to being based on the idea of tracing so called "atoms"—components of N-atomic measures μ—instead of moments of these measures. The number $N \geq 1$ can be chosen relatively freely based on the essential