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Randomized Approximations of the Image Set of Nonlinear Mappings with Applications to Filtering

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Abstract: The aim of this paper is twofold: In the first part, we leverage recent results on scenario design to develop randomized algorithms for approximating the *image set* of a nonlinear mapping, that is, a (possibly noisy) mapping of a set via a nonlinear function. We introduce minimum-volume approximations which have the characteristic of guaranteeing a low probability of violation, i.e., we admit for a probability that some points in the image set are not contained in the approximating set, but this probability is kept below a pre-specified threshold ε. In the second part of the paper, this idea is then exploited to develop a new family of randomized prediction-corrector filters. These filters represent a natural extension and rapprochement of Gaussian and set-valued filters, and bear similarities with modern tools such as particle filters.

Keywords: Randomized algorithms, filtering, nonlinear systems, semialgebraic sets.

1. INTRODUCTION

In recent years, randomized algorithms have gained increasing popularity in the field of control of uncertain systems; e.g., see Calafiore et al. [2011], Tempo et al. [2013], due to their ability of dealing with large and complex uncertainty structures, thus extending the applicability of the robust control methods. This is obtained by shifting from the robustness paradigm, where one looks for *guaranteed* performance which should hold for every possible instance of the uncertainty, to an approach where *probabilistic guarantees* are accepted, i.e., performance is guaranteed only within a given level of probability $\varepsilon > 0$. The main technical tool that permits to obtain computationally tractable solutions are randomized algorithms, which could be seen as extensions of the classical Monte Carlo method, and are based on the extraction of random samples of the uncertainty.

In this paper, we exploit these ideas for finding reliable approximations of the image of a set through a nonlinear mapping, which we refer to as the *image set*. This set is in general nonconvex (possibly not connected), so that classical approximations may be rather difficult to compute and in general may turn out to be very conservative.

The first part of the paper adapts and significantly extends recent results presented by some of the authors in Dabbene et al. [2010], where a new definition of "goodness" of approximation was provided in probabilistic terms. Namely, an approximating set $\mathcal{A}$ of a set $\mathcal{X}$ is deemed to be "good" if it contains "most" of the points in $\mathcal{X}$ with high probability. Contrary to classical approximating sets, which are generally either *outer* or *inner*, the ensuing approximating set "optimally describes" the set without neither containing nor being contained in it. This new concept allows to obtain generally tighter approximations, providing a probabilistic characterization of the set, which is particularly appealing in many contexts (even if it may not be desired in others, such e.g. as safety analysis). We recall that outer bounding sets, that is, approximations which are guaranteed to contain the set $\mathcal{X}$, have been very popular in the set-membership approach, and have been used in designing set-theoretic state estimators for uncertain discrete-time nonlinear dynamic systems; e.g., see El Ghaoui and Calafiore [2001], Alamo et al. [2008]. Inner ellipsoidal approximations have been introduced for instance in the context of nonlinear programming problems [Nesterov and Nemirovski, 1994] and tolerance design [Wojciechowski and Vlach, 1993].

In Sections II and III, the results of Dabbene et al. [2010] are particularized to the specific problem at hand, and are also generalized by considering a new family of approximating sets which are based on the construction of minimum volume *polynomial superlevel sets*, recently introduced in Dabbene and Henrion [2013]. With these approximating sets, the original convexity requirement can be relaxed, since they can be nonconvex/nonconnected, thus allowing for better descriptions.

The second part of the paper extends these ideas to the design of probabilistic filters for nonlinear discrete time dynamical systems subject to random uncertainty. In

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particular, the image set approximation is used to design a probabilistically optimal predictor filter. The rationale behind this approach is to "describe" the state position at step k by a set where the state lies with high probability. The prediction step is then combined with a correction step where the propagated set is trimmed based on the available measurements at time k and on the measurement noise assumptions. A detailed discussion on the proposed randomized prediction correction filter is given in Section IV, where a discussion on the related literature is also reported. Numerical examples conclude the paper.

Notation: For a symmetric matrix P, $P \succ 0$ means that P is positive definite. We denote by $\mathcal{B}_p$ the unit ball in the ℓ_p norm: $\mathcal{B}_p \doteq \{z \in \mathbb{R}^n : \|z\|_p \leq 1\}$. The volume or, more precisely, the Lebesgue measure of a compact set $\mathcal{X}$ is denoted by $\mathrm{vol}\,\mathcal{X} \doteq \int_{\mathcal{X}} \mathrm{d}x$. The uniform measure over $\mathcal{X}$ is denoted by $\lambda_{\mathcal{X}}$, i.e. $\lambda_{\mathcal{X}}$ is such that, for any set $\mathcal{Y} \subseteq \mathcal{X}$, $\lambda_{\mathcal{X}}(\mathcal{Y}) = \mathrm{vol}\,\mathcal{Y}/\mathrm{vol}\,\mathcal{X}$. The set of all polynomials of order less than or equal to σ is denoted by $\mathbb{P}_\sigma$. The monomial basis for this set is represented by the (column) vector $\pi_\sigma(x)$ and any polynomial $q \in \mathbb{P}_\sigma$ can be expressed in the form $q(x) = \pi_\sigma^\top(x) q = \pi_{\lceil\sigma/2\rceil}^\top(x) Q \pi_{\lceil\sigma/2\rceil}(x)$ where q is a vector and Q is a symmetric matrix of appropriate dimensions, referred to as the Gram matrix.

2. MINIMUM VOLUME APPROXIMATIONS

Consider the following nonlinear mapping:

$$x_+ = f(x, w) \tag{1}$$

which represents the one-step evolution of a discrete-time dynamical system, with $x, x_+ \in \mathbb{R}^n$ representing the current and future states, respectively, and $w \in \mathbb{R}^{n_w}$ describing a process noise vector. We assume that the current state is confined within a compact set $\mathcal{X} \subset \mathbb{R}^n$ and that the noise w also belongs to a given compact set $\mathcal{W}$. The problem we are interested in is to find a good approximation to the set of points that can be obtained from (1) by starting from $x \in \mathcal{X}$ and accounting for all possible values of the noise $w \in \mathcal{W}$, that is, find an approximation of the *image set* defined as

$$\mathcal{X}_+ \doteq f(\mathcal{X}, \mathcal{W}) = \{x_+ \in \mathbb{R}^n \colon \exists x \in \mathcal{X}, w \in \mathcal{W} \colon x_+ = f(x, w)\}.$$

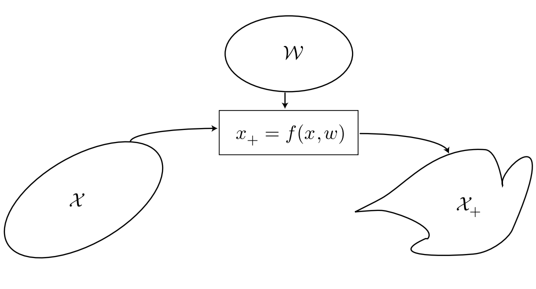

Fig. 1. Set image of $\mathcal{X}$ with noise $w \in \mathcal{W}$ under map (1) .

Motivated by the computational complexity issues and the fact that deterministic formulations of the approximation problem may not be suitable in practical situations, in Dabbene et al. [2010] the authors proposed an original approach for tackling the problem, based on a probabilistic viewpoint. To this end, a probabilistic information over $\mathcal{X}$ and $\mathcal{W}$ is assumed to be known (which is actually the case in many applications), and an approximation $\mathcal{A}$ of $\mathcal{X}_+$ is deemed to be "good" if it contains all points in $\mathcal{X}_+$ with high probability. More formally, we assume that the sets $\mathcal{X}$ and $\mathcal{W}$ are endowed with probability measures $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{W}}$. Then, for a given *reliability level* ε, the following concept of ε-probabilistic approximation is introduced.

Definition 1. (ε-probabilistic approximation of $\mathcal{X}_+$). The set $\mathcal{A}$ is an *ε-probabilistic approximation of the set $\mathcal{X}_+$* if

$$\mathrm{Viol}(\mathcal{A}) \leq \varepsilon,$$

with

$$\begin{aligned}\mathrm{Viol}(\mathcal{A}) &\doteq \mathsf{Pr}\{x \in \mathcal{X}, w \in \mathcal{W} \colon x_+ = f(x, w) \notin \mathcal{A}\} \\ &= \int_{\mathcal{X}_+ \setminus \mathcal{A}} f(x, w) \mathrm{d}\mu_{\mathcal{X}}(x) \mathrm{d}\mu_{\mathcal{W}}(w).\end{aligned} \tag{2}$$

Note that the probability in (2) is measured with respect to the underlying measures $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{W}}$. The left-hand side of the equation is referred to as the *violation probability of the set $\mathcal{A}$*.

The main characteristic of this approach is that an ε-probabilistic approximating set has neither to cover nor to be fully contained in $\mathcal{X}$; it just has to guarantee that the violation probability of the set $\mathcal{A}$ is bounded by ε. Clearly, we are interested in finding the smallest among such sets. In the sequel, we first define the two families of approximating sets considered in this paper.

2.1 Convex approximating sets: Ellipsoids, parallelotopes, and hyperrectangles

The following general description of the *norm-based approximating set* (NAS) was introduced in Dabbene et al. [2010]:

$$\mathcal{A}(c, P) \doteq \{x \in \mathbb{R}^n \colon \|P(x - c)\|_p \leq 1, P = P^\top \succeq 0\}. \tag{3}$$

Note that the family of sets above is parameterized by the positive-definite *shape matrix* P, and by the center $c \in \mathbb{R}^n$; it represents a generalization of the classical ellipsoidal set for norms different from the Euclidean one.

Indeed, for $p = 2$, we obtain the ellipsoid

$$\mathcal{E}(c, P) \doteq \{x \in \mathbb{R}^n \colon x = c + P^{-1} z, \|z\|_2 \leq 1\},$$

and for $p = \infty$ we get a so-called *elementary parallelotope*, a special-type polytope with parallel faces (these can be viewed as a particular class of zonotopes [Alamo et al., 2005] with positive definite generator matrix). In particular, if P is chosen to be diagonal, we obtain a classical hyper-rectangle.

It follows that, in general, the problem of finding the *minimum volume* NAS containing $\mathcal{X}_+$ can be rewritten in the form of the following robust convex problem:

$$\begin{aligned} &\min_{P,c} \log\det(P^{-1}) \\ &\quad \text{s.t.} \quad P = P^\top \succ 0, \\ &\qquad\quad \|Px_+ - c\|_p \leq 1 \ \forall x_+ \in \mathcal{X}_+. \end{aligned} \tag{NAS-robust}$$

2.2 Nonconvex approximations: Polynomial superlevel sets

In Dabbene and Henrion [2013], nonconvex set approximations based on the superlevel set of a multidimensional polynomial have been introduced, and shown to represent a simple and efficient way for describing complex shaped sets. Formally, assume we are given a compact semialgebraic set

$$\mathcal{S} := \{x \in \mathbb{R}^n : b_i(x) \geq 0, i = 1, 2, \ldots, m_b\}$$

such that $\mathcal{X}_+ \subseteq \mathcal{S}$, with $b_i(x)$ being given polynomials (the set $\mathcal{S}$ is usually a hyper-rectangle). Then, given a