Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-14 (2015) 113–119

Application of Facial Reduction to H_∞ State Feedback Control Problem ⋆

Hayato Waki * **Noboru Sebe** **

* *Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku Fukuoka 819-0395, Japan (e-mail: waki@imi.kyushu-u.ac.jp)*
** *Department of Artificial Intelligence, Faculty of Computer Science and Systems Engineering Kyushu Institute of Technology, 680-4 Kawazu, Iizuka-shi, Fukuoka 820-8502, Japan (e-mail: sebe@ai.kyutech.ac.jp)*

Abstract: One often encounters numerical difficulties in solving linear matrix inequality (LMI) problems obtained from H_∞ control problems. We discuss the reason from the viewpoint of optimization. It is empirically known that a numerical difficulty occurs if the resulting LMI problem or its dual is not strongly feasible.
In this paper, we provide necessary and sufficient conditions for LMI problem and its dual not to be strongly feasible, and interpret them in terms of control system. For this, facial reduction, which was proposed by Borwein and Wolkowicz, plays an important role. We show that a necessary and sufficient condition closely related to the existence of invariant zeros in the closed left-half plane in the system, and present a way to remove the numerical difficulty with the null vectors associated with invariant zeros in the closed left-half plane. Numerical results show that the numerical stability is improved by applying it.

Keywords: H_∞ control, linear matrix inequalities, state feedback control, facial reduction, invariant zeros

1. INTRODUCTION

H_∞ control problems have attracted attention from a lot of researchers in control and optimization fields since primal-dual interior-point methods were proposed in 90's. H_∞ control problems can be reformulated as Linear matrix inequalities (LMI) problem and be efficiently solved by LMI software, such as SeDuMi (Sturm (1999)), SDPT3 (Toh et al. (1999)) and SDPA (Yamashita et al. (2003)), etc. Still, one often encounters some numerical difficulties in solving LMI problems obtained from H_∞ control problems by these LMI software.

The purpose of this paper is to investigate the reason why solutions and values obtained from LMI software are often inaccurate to a fixed tolerance. It is empirically known that when either an LMI problem or its dual is not strongly feasible, the numerical instability will occur. In that case, optimal solutions may not exist. In addition, primal-dual interior-point methods may not converge numerically. In fact, theoretical results of primal-dual interior-point methods is required strongly feasible for both LMI problems and its dual.

Facial Reduction (FR) is useful in such cases. It can detect whether a given LMI problem and its dual are strongly feasible or not. In addition, if not so, FR finds a certificate and generates a pair of LMI problem and its dual which are strongly feasible and equivalent to the original pair. However, the execution of FR needs much more computation cost than solving the original LMI problem.

In this paper, we deal with state feedback controls for linear time invariant systems. We present necessary and sufficient conditions for LMI problems and its dual of H_∞ control problems obtained from them not to be strongly feasible. One of them is related to the existence of a stable invariant zero in a given system. In other words, the dual problem is not strongly feasible if the system has a stable invariant zero.

We also provide how to remove the numerical difficulty caused by non-strong feasibility. This is also based on FR and it generates a smaller LMI problem by using invariant zeros in $\overline{\mathbb{C}_-}$. Interestingly, the resulting LMI problem can be obtained from a subsystem of the closed loop system. In fact, a non-singular matrix used in the reduction consists of invariant zeros in $\overline{\mathbb{C}_-}$ and plays an important role in FR for a given LMI problem. We show that the subsystem is obtained by applying the transformation with the matrix into the closed loop system. This implies that one can retrieve a state feedback gain for the closed loop from one for the subsystem. We also present numerical experiments to see the improvement on the numerical stability.

1.1 Literature related to this topic

Balakrishnan and Vandenberghe (2003) applies FR to the analysis and design of H_2 state feedback control, and

⋆ The first author was supported by JSPS KAKENHI Grant Numbers 22740056 and 26400203.

10.1016/j.ifacol.2015.09.443

showed the relationship between the strong feasibility and invariant zeros in H_2 state feedback control. Vandenberghe et al. (2005) proposes a fast implementation for LMI problems based on KYP lemma. This approach reformulates the dual of a given LMI problem into the form of the LMI problem. Although this implementation reduces the computational complexity, this is essentially different from FR. In fact, this does not change the strong feasibility of the resulting LMI problem.

The effect of invariant zeros for the performance index γ in H_∞ feedback control has been already discussed in Stoorvogel (1991), Scherer (1992a) and Scherer (1992b). We provide similar results by applying FR. This implies that we should reduce LMI problems from the view point of numerical accuracy, as far as using software based on PDIPMs. Control systems whose D_{12} is not full column rank has been also handled in this paper as well as those papers. We can verify by using FR that the optimal performance index γ for such a system is equivalent to γ for a system in which a differentiator is added.

1.2 Notation and symbols

Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real and complex numbers, respectively. We represent the sets of complex numbers with nonnegative real parts and nonpositive real parts by $\overline{\mathbb{C}_+}$ and $\overline{\mathbb{C}_-}$, respectively. Let $\mathbb{R}^n$ be the set of n-dimensional Euclidean space. Let $\mathbb{R}^{m\times n}$, $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ be sets of $m\times n$ real matrices, $n\times n$ symmetric matrices, $n\times n$ positive semidefinite matrices and $n\times n$ positive definite matrices, respectively. We denote the $m\times n$ zero matrix by $O_{m\times n}$. For $A, B \in \mathbb{R}^{m\times n}$, we define $A \bullet B := \mathrm{Trace}(AB^T)$. For $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}(A) = A + A^T$.

2. PRELIMINARY

2.1 Linear matrix inequality and its strong duality theorem

Linear Matrix Inequality (LMI) problem is formulated as follows:

$$\inf_{x\in\mathbb{R}^m, X\in\mathbb{S}^n} \left\{ c^T x : X = \sum_{j\in\mathcal{M}} x_j F_j - F_0, X \in \mathbb{S}^n_+ \right\}, \quad (1)$$

where $c \in \mathbb{R}^m$, $\mathcal{M} = \{1, \ldots, m\}$ and $F_0, \ldots, F_m \in \mathbb{S}^n$. We denote the optimal value of (1) by θ^*_P. Throughout this paper, we assume that $F_1, \ldots, F_m$ is linearly independent. One can obtain an approximation of an optimal solution of (1) to any given tolerance by applying Primal-Dual Interior-Point Method (PDIPM). In fact, many variants of PDIPM are proposed and implemented as optimization software in SeDuMi (Sturm (1999)), SDPT3 (Toh et al. (1999)), SDPA (Yamashita et al. (2003)) etc.

The dual problem of (1) can be formulated as follows:

$$\sup_{Y\in\mathbb{S}^n} \left\{ F_0 \bullet Y : F_j \bullet Y = c_j \ (j \in \mathcal{M}), Y \in \mathbb{S}^n_+ \right\}. \quad (2)$$

We denote the optimal value of (2) by θ^*_D. It is well-known that the strong duality theorem holds for (1) and its dual (2) under a mild assumption which is called *Slatter condition.* See Theorem 1 below. Unlike to Linear Program (LP), the strong duality theorem for (1) and (2) requires the Slatter condition. Problem (1) is said to be *strongly feasible* if there exists $(\hat{x}, \hat{X}) \in \mathbb{R}^m \times \mathbb{S}^n$ such that $\hat{X} = \sum_{j\in\mathcal{M}} \hat{x}_j F_j - F_0, \hat{X} \in \mathbb{S}^n_{++}$. Similarly, (2) is said to be *strongly feasible* if there exists $\hat{Y} \in \mathbb{S}^n_{++}$ such that $F_j \bullet \hat{Y} = c_j$ for all $j \in \mathcal{M}$. Slatter condition for (1) is satisfied if (1) is strongly feasible. Slatter condition for (2) is defined, similarly.

Theorem 1. (Renegar (2001)) If (2) is strongly feasible and (1) is feasible, then $\theta^*_P = \theta^*_D$ and (1) has an optimal solution. Similarly, if (1) is strongly feasible and (2) is feasible, then $\theta^*_P = \theta^*_D$ and (2) has an optimal solution.

It should be noted that Slatter conditions for both (1) and its dual (2) guarantee the convergence of PDIPM. See Renegar (2001) and references therein for more details.

2.2 Facial reduction for LMI problem (1) and its dual (2)

Borwein and Wolkowicz (1981) propose an approach to the strong duality theorem for convex optimization problems without assuming Slatter condition and any constraint qualifications. The approach is called *Facial Reduction (FR).* Ramana (1997) and Ramana et al. (1997) discuss the application of FR into LMI problem (1) and its dual (2). Some extensions are discussed in Pataki (2013) and Waki and Muramatsu (2013).

Facial Reduction Algorithm (FRA) is an algorithm that generates an LMI problem which is strongly feasible by using a given LMI problem, or that detects the infeasibility. FRA has the property of a finite convergence. In each iteration, FRA finds a nonzero solution of a problem that consists of LMI or detects the infeasibility. Since one has to solve a similar LMI problem in each iteration, the computation spends as much cost as solving the original LMI problem. Moreover, FRA requires an exact solution of a generated problem in each iteration, and thus FRA is not practical algorithm from the viewpoint of computational practice.

Still, when (1) and/or its dual (2) is not strongly feasible, one often encounters numerical difficulty on solving them. See Henrion and Lasserre (2005), Navascués et al. (2013), Waki et al. (2012), Waki (2012) and Waki and Muramatsu (2013) for more details. Hence, it is necessary to apply FRA without solving problems that consist of LMI.

To propose such an approach, we provide necessary and sufficient conditions in Theorem 2 by using FR that (1) and its dual (2) are not strongly feasible. In the next section, we will apply Theorem 2 into LMI problems obtained from H_∞ state feedback control problems.

Theorem 2. Problem (1) is not strongly feasible if and only if there exists a nonzero $\hat{Y} \in \mathbb{S}^n$ such that

$$F_j \bullet \hat{Y} = 0 \ (j \in \mathcal{M}), F_0 \bullet \hat{Y} \geq 0 \text{ and } \hat{Y} \in \mathbb{S}^n_+. \quad (3)$$

In particular, if $\hat{Y}$ satisfies $F_0 \bullet \hat{Y} > 0$, then (1) is infeasible. Moreover, (1) is equivalent to the following problem:

$$\inf_{x\in\mathbb{R}^m, X\in\mathbb{S}^n} \left\{ c^T x : X = \sum_{j\in\mathcal{M}} x_j F_j - F_0, X \in \mathbb{S}^n_+ \cap \{\hat{Y}\}^\perp \right\}, \quad (4)$$

where $\{\hat{Y}\}^\perp$ denotes the subspace $\{X \in \mathbb{S}^n : X \bullet \hat{Y} = 0\}$ of $\mathbb{S}^n$. Similarly, (2) is not strongly feasible if and only if