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Relational measurements and uncertainty

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ABSTRACT

In representational measurement theory, the current theory of all measurements, calibration and sampling processes are assumed to be a linear transformation of the coordinate system, of no effect. In this paper calibration and sampling are shown to be independent non-linear processes which do change measurement results. Relational measurement theory is developed to include calibration and sampling. The measurement changes caused by calibration and sampling are proven to be equal to the quantum measurement disturbance described by the universal uncertainty relation which has been verified by experiments. Therefore relational measurement theory explains the measurement disturbance in quantum mechanics.

1. Introduction

In representational measurement theory [1] a measurement result is a magnitude of equal intervals. In this paper relational measurement theory [2] defines a *measurement result* as a sum of intervals, where each interval's magnitude is modified by both calibration and sampling processes. The application of relational measurement theory to quantum systems explains the disturbance of one observable when measuring a second observable [3]. In this paper: *sampling* is the division of a continuous observable into discrete intervals [4], a *measurement* is the sum of the projection of all these discrete intervals onto the measuring apparatus intervals [5], and *calibration* is the correlation of sets of the measuring apparatus intervals to externally defined intervals [6]; together these three processes produce a measurement result.

In von Neumann's development of a measurement in Hilbert space [7], the possible discrete elements are equivalent (not necessarily equal) and of varying probability. Each such quantum element (in one dimension) may be seen as one of a measuring apparatus' minimum intervals in the same dimension. In representational measurement theory, these elements/intervals are assumed to be equal. Then the standard deviation of a distribution of these elements/intervals is zero.

The Heisenberg Uncertainty Relation (HUR) [8] identifies an inconsistency when the elements/intervals of a measurement system are assumed to be equal. In this paper the standard deviations in the HUR, shown in Eq. (7), are proved to be of the magnitudes of the observable's intervals, not of the magnitude of the observable, and are never zero. In the relational measurement view of the HUR, each observable's standard deviation is a measure of a distribution of calibration corrections. These calibration corrections appear as a measurement disturbance of the second observable when both observables are correlated by calibration corrections to the same reference.

Sections 5 develop relational measurement theory by applying concepts from classical metrology. Sections 6 and 7 apply relational measurement theory to the HUR, proving that the standard deviation of a distribution of discrete intervals in the HUR equals the effect of calibration on experimental measurement results.

In 2003, Ozawa [3] developed the universal uncertainty relation, Eq. (12), which is shown to support relational measurement theory. His universal uncertainty relation, with a minor modification, formalizes both calibration and sampling indeterminacy [9]. In Appendix A, experiments by others [10] are presented which verify the universal uncertainty relation and therefore relational measurement theory.

2. Relational measurements

A *relational measurement system* defines the *mean measurement result* as the product of measurement magnitude and the standard deviation of the measurement intervals due to calibration and sampling processes (4).

Fig. 1 identifies m intervals of i_u, a relational measurement result, relative to m intervals of i, a representational measurement. Sampling, often treated as part of a measurement process, divides the observable into increments (Δs) indicated in Fig. 1 by short

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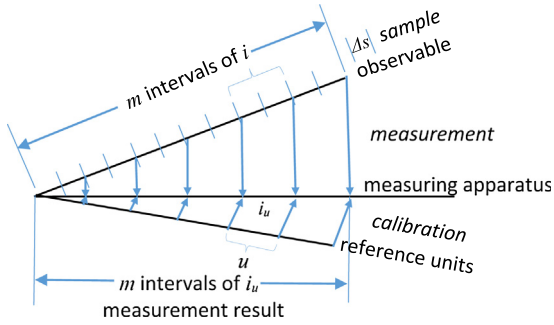

Fig. 1. Relational measurement system.

lines at right angles to the observable. A measurement is indicated by downward arrows which project the intervals (usually two or more sample increments) onto the measuring apparatus to be summed. Calibration of the measuring apparatus intervals to u (a reference unit) is indicated by the inward arrows. Calibration is represented in an orthogonal plane as it is independent of sampling and measurement. Both planes have a common basis, the measuring apparatus.

Example: X is a rod's length (the observable). X is experimentally defined as a magnitude (m) of intervals each correlated to a minimum reference unit x (in this example a centimeter). The magnitude of each correlated interval is i_x. The representational measurement (normalized) magnitude of the intervals of X is $i = 1/m$. The experimental measurement result precision [11] (local variation) is $\pm\Delta s_{xi}$ per interval i [12]. The measuring apparatus is a meter scale which defines the magnitude of the intervals, where each interval is calibrated to x, the reference interval. A calibration process makes feasible comparisons between independent measurement results. The measured accuracy [11] (relative variation) of each i_x is determined by calibration to x. The differences between mi, mx and mi_x are not treated in representational measurement theory.

Applying Fig. 1, Δx_i = calibration variation of each interval and Δs_{xi} = sample increment of each i interval, the calibration (1) and sampling (2) operators are:

$$x \pm \Delta x_i = i_x \tag{1}$$

$$i \pm \Delta s_{xi} = i_x \tag{2}$$

The indeterminacy (the sum of calibration and sampling variation) of i_x is:

$$\Delta i_x = (\mp\Delta x_i) + (\pm\Delta s_{xi}) \tag{3}$$

In representational measurement theory, this indeterminacy is assumed to reduce to zero as accuracy and precision move toward perfect [13]. This is not experimentally possible. In all discrete measurements of continuous observables there is a non-cancelable minimum interval indeterminacy $min\,\Delta i_x <$ the product of the speed of light and the reciprocal of the highest sampling frequency observed. The higher the sampling frequency, the smaller $|min\,\Delta i_x|$ (the vertical bars represent an absolute value). When all other indeterminacy is nulled, the magnitude of each interval i_x randomly deviates by just less than $\pm\Delta s_{xi}$, the sampling increment. Therefore Δx_i cannot completely cancel Δs_{xi}.

Eq. (4) presents four different functions which represent the observable $\langle X\rangle$ (where brackets indicate the mean of X) based upon three different assumptions.

$$\int \psi^* \widehat{X} \psi dx \to \sum_{i=1}^{i=m} i \to \sum_{i_x=1}^{i_x=m} i_x \cong m[\sigma(i_x) + \sigma(\Delta s_{xi})] = \langle X\rangle \tag{4}$$

The first function is from quantum mechanics [14]: $\int \psi^* \widehat{X} \psi dx$. ψ^* is the complex conjugate of the state vector ψ (observable) of x and $\widehat{X}$ is the operator of X, therefore $\int \psi^* \widehat{X} \psi dx$ (assumes infinitesimal sampling increments) represents the mean of a continuous observable. Normalized sampling (assumes $i = x$) transforms $\int \psi^* \widehat{X} \psi dx$ to a discrete measurement [15], shown as the second function: $\sum_{i=1}^{i=m} i$ which is the rod's length, m intervals of i.

Calibration (1) transforms $\sum_{i=1}^{i=m} i$ to the third function: $\sum_{i_x=1}^{i_x=m} i_x$ which is the rod's length in i_x intervals and is the current classical metrology model [16]. When the sampled increment $\Delta s_{xi} \ll i_x$ (assumed in classical metrology), $\sum_{i_x=1}^{i_x=m} i_x$ is close to the fourth function: $m[\sigma(i_x) + \sigma(\Delta s_{xi})]$ as $\sigma(\Delta s_{xi}) < 1$ sampled increment (derived below in Section 3).

The fourth function, $m[\sigma(i_x) + \sigma(\Delta s_{xi})]$, does not require these three assumptions. This function sums over the common basis both the calibration (1) and sampling (2) operators and is the only function of the four which represents the mean of the experimental measurement results of a continuous observable at all experimentally possible sampling frequencies.

3. Sampling and calibration examples

Consider a digital voltmeter (measuring apparatus) where 00.01 is the voltmeter's display of the minimum interval. Measuring a fixed voltage (observable) multiple times produces a stochastic distribution of measurement results. The maximum indeterminacy of this distribution is specified by the manufacturer, for all voltmeters of this model, to be ±1% indeterminacy. To maintain the ±1% of a 0.01 V measurement requires ±0.0001 V precision, i.e., each interval is between 0.0099 and 0.0101 V. The ±1% indeterminacy allows a laboratory with multiple voltmeters to make comparable measurements or compare measurements with other laboratories. To achieve this precision, a sampling increment of 0.0001 (Δs_{xi}) V or less is required. The precision of a measurement result is ultimately limited by the Planck constant, the minimum possible sampled increment, and is never zero.

A stochastic distribution of voltmeter measurement results occurs when applying the 1.00000 V observable (continuous relative to the sampling increment of 0.0001 V) to many of the same model voltmeters. Indeterminacy of the continuous observable less than the sampling increment is not identifiable. The sampling process causes each of the 100 (m) 0.01 intervals to have an indeterminacy of $\Delta s_{xi} = -0.0001$, 0.0000 ($<\Delta s_{xi}$) or, +0.0001. Less than Δs_{xi} is not a zero state, but the transition between $+\Delta s_{xi}$ and $-\Delta s_{xi}$. Therefore $<\Delta s_{xi}$ statistically occurs less often than either $\pm\Delta s_{xi}$ and the standard deviation of this sampling distribution is always $< |\Delta s_{xi}|$.

In this sampling example, the three possible interval magnitudes (from (2)) are: $i \pm \Delta s_{xi} = i_x = 0.0099i$, $0.0100i$ or $0.0101i$. This identifies 3^{100} combinations of the 100 intervals which establish the probability of the 200 valid measurement results with values between 0.9900 to 1.0100 and within the defined ±1% precision. The distribution of the 3^{100} possible combinations of the 100 intervals will, as the number of voltmeter measurements increase, converge to a normal distribution (bell shaped curve) as described by the central limit theorem. Such a distribution (i.e., sampling noise) occurs in all measurement results and is sometimes identified in the literature as $1/f$ noise, where f represents the sampling frequency in [17].

As a further example, an observable's angular rotation is determined by counting the teeth of an attached gear with m teeth per 2π rotation. When counted by an observer, the gear teeth are assumed to be intervals of equal width and no indeterminacy appears. But a perfect measurement of angular rotation by using