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Blending for $\mathcal{H}_\infty$ performance: a group theoretic approach

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Abstract: In order to design efficient algorithms that work on the set of controllers that fulfill a given property, e.g., stability or a norm bound, it is important to have an operation that preserves that property, i.e., a suitable blending method. Concerning stability, a traditional approach is to use the Youla parametrization and the corresponding parameters as a starting point. While this method guarantees stabilizability as the invariant property for the fairly large class of strictly proper plants, there are also other solutions to the problem.
The authors already provide a detailed analysis for feedback stability placing the controller blending problem in a general setting by pointing to the basic global geometric structures that are related to well-posedness and feedback stability. In this paper these efforts are continued and the group structure corresponding to performance problems, e.g., those related to a suboptimal $\mathcal{H}_\infty$ design, is presented. Besides its educative value the presentation provides a possible tool for the algorithmic development.

Keywords: controller parametrisation; controller blending; performance guarantee

1. INTRODUCTION

Felix Klein, in the late 1800s, developed an axiomatic basis for Euclidean geometry that started with the notion of an existing set of transformations and he proposed that geometry should be defined as the study of transformations (symmetries) and of the objects that transformations leave unchanged, or invariant. The set of symmetries of an object has a very nice algebraic structure: they form a group. By studying this algebraic structure, we can gain deeper insight into the geometry of the figures under consideration.

A common tool in formulating robust feedback control problems is to use system interconnections that can be described as linear fractional transforms (LFTs), as a general framework to include the rational dependencies that occur. In this context a special role plays the intimate relationship between Möbius transformations and LFTs. It is known that the set of stabilizing controllers for a given plant and the set of all suboptimal $\mathcal{H}_\infty$ can be expressed by using certain Möbius transformations and LFTs, respectively.

In Szabó et al. [2014] the authors emphasise Klein's approach to geometry and demonstrate that a natural framework to formulate different control problems is the world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms. The observation that any geometric property of a configuration, which is invariant under an euclidean or hyperbolic motion, may be reliably investigated after the data has been moved into a convenient position in the model, facilitates considerably the solution of the problems.

In contrast to traditional geometric control theory, see, e.g., Wonham [1985], Basile and Marro [2002] for the linear and Isidori [1989], Jurdjevic [1997], Agrachev and Sachkov [2004] for the nonlinear theory, which is centered on a local view, this approach provides a global view. While the former uses tools from differential geometry, Lie algebra, algebraic geometry, and treats system concepts like controllability, as geometric properties of the state space or its subspaces the latter focuses on an input-output – coordinate free – framework where different transformation groups which leave a given global property invariant play a fundamental role.

In the first case the invariants are the so-called invariant or controlled invariant subspaces, and the suitable change of coordinates and system transforms (diffeomorphisms), see, e.g., the Kalman decomposition, reveal these properties. In contrast, our interest is in the transformation groups that leave a given global property, e.g., stability or $\mathcal{H}_\infty$ norm, invariant. One of the most important consequences of the approach is that through the analogous of the classical geometric constructions it not only might gave hints for efficient algorithms but the underlaying algebraic structure, i.e., the given group operation, also provides tools for controller manipulations that preserves the property at hand, called controller blending.

There are a lot of applications for controller blending: both in the LTI system framework, Niemann and Stoustrup [1999], Stoustrup [2009] and in the framework using gain-scheduling, LPV techniques, see Shin et al. [2002], Chang

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and Rasmussen [2008], Hencey and Alleyne [2008, 2010]. These approaches exploits the Youla parametrization of stabilizing controllers. As it was shown in Szabó and Bokor [2015], this approach does not provide an exhaustive characterization of the topic because the mere addition is suitable only if the plant (or the controller) is strictly proper. Moreover, one can define a blending that preserves stability and it is defined directly in terms of the plant and controller, without the necessity to use any factorization. As an interesting side effect of these investigations it turns out that the proposed operation leaves invariant the strongly stabilizing controllers and defines a group structure on them, too.

The starting point of this paper is the fact that solutions of the quadratic performance problems, e.g., a suboptimal $\mathcal{H}_\infty$ design, are parametrized by the elements of the unit ball. On the unit ball, however, we cannot define directly an operation in a trivial way that bears a nice algebraic structure. The group actions that correspond to the addition of stable plants seen for the Youla parametrization are the hyperbolic motions of the unit ball, determined by the J-unitary operators. The paper presents in details an explicit parametrization of these operators and the corresponding induced operation on this parameter space. These results provides a starting point in developing algorithms that uses some sort of iteration. Due to the increase in the plant order, the potential targets are analysis problems, rather than design applications.

Concerning the structure of the presentation, Section 2 gives a more detailed motivation background for the problem tackled in the paper. For the sake of completeness in Section 3 we summarize the basic results of Szabó and Bokor [2015] related to the feedback stability problem: we present the operations on controllers that leave invariant stability and point out the difference between the direct method, when we directly blend the controllers, and the indirect method, when the blending is done on the parameters of a suitable representation (Youla). Section 4 presents the main result of the paper and provides a parametrization of the J-unitary matrices and the group operation of this parameter space that corresponds to the hyperbolic motions defined by these J-unitary matrices. As such, the presented results provides a counterpart of the indirect approach for the case of performance problems. We conclude the paper by indicating some further research directions.

2. MOTIVATION AND PROBLEM SETUP

2.1 The $\mathcal{H}_\infty$ problem

The most typical robust performance problem can be cast as a suboptimal normalized $\mathcal{H}_\infty$ design, where for a given generalized plant description P we seek all controllers K that internally stabilize the loop and achieves $\|\mathfrak{F}_l(P,K)\| < 1$. It is a standard fact that by applying the Youla parametrization the performance can be expressed in affine terms as $\mathfrak{F}_l(\bar{P},Q)$, with the stable matrix $\bar{P} = \begin{pmatrix} n_{zw} & n_{zu} \\ \tilde{n}_{yw} & 0 \end{pmatrix}$.

With a further simplification, i.e., an inner(co-inner)-outer factorization we can consider a parametrization where n_{zu} and n_{yw} are isometries. Then we have the invariance relation $\|\mathfrak{F}_l(\bar{P},Q_1) - \mathfrak{F}_l(\bar{P},Q_2)\| = Q_1 - Q_2$ of the Euclidean distance. However, this is not the invariance we are interested in.

Since the solutions of the suboptimal $\mathcal{H}_\infty$ design are parametrized by the elements of the unit ball, the relevant distance here is the hyperbolic one. There are a lot of different possibilities to arrive to this conclusion. One of the most well-known approaches assumes either left or right invertibility of P and uses the scattering framework by augmenting the plant, if necessary, to obtain a well defined Potapov-Ginsburg transform $\hat{P}$, see Ball et al. [1991], Kimura [1997] for details. Then a J-inner outer factorization $\hat{P} = \hat{\Theta}_a\hat{R}$, with a block tridiagonal structure of the outer factor that corresponds to the structure of the augmentation, solves the problem. The controllers are given by the Möbius transform $\mathfrak{M}_{\hat{R}^{-1}}(H_a)$ with

$$H_a = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}, \quad \|H\| < 1,$$

while the closed loop is given by $\mathfrak{M}_{\hat{\Theta}_a}(H_a)$. Recall that Θ_a is an inner function, thus

$$\mathfrak{F}_l(P,K) = \mathfrak{M}_{\hat{\Theta}_a}(H_a) = \mathfrak{F}_l(\Theta_a, H_a) < 1. \tag{1}$$

For the details on J-inner and J-lossless functions see Dym [1989] and Kimura [1997].

An other approach, see Green et al. [1990], Green [1992], instead of augmentation uses two J-spectral factorizations. In those papers the characterization of the close loop is implicit. However, the same idea appears in a more transparent way in Tsai et al. [1993], Tsai and Gu [2014], where by a slight extension of the traditional scattering approach one can obtain the more transparent description of the solution of an $\mathcal{H}_\infty$ problem: the controller is expressed as a Möbius transform of a unit ball generated by a unimodular (outer) operator while the performance is expressed as the image of this unit ball under the action of two Möbius transforms determined by J-isometries.

These facts motivates our interest in the unit ball: if we would like to blend controllers and guarantee a prescribed performance level, we should blend elements of the unit ball. One possible approach is to consider the action of the J-unitary operators on this ball – they obviously form a group considering the composition of operators– and to express the desired operation as a group homomorphism. This is the same idea (the indirect approach) that we follow with the addition of the Youla parameters to blend stable controllers. To fulfill this program a suitable parametrization is needed that relates the J-unitary operators to the elements of the unit ball. This task is completed in the present paper.

A further application field results when we consider the parametrization of the solution of different quadratic performance problems by using a state space description and LMI techniques. All these sets are parametrized by elements of a matrix unit ball, see Szabó et al. [2012a,b, 2013]. For an application for the design of stable $\mathcal{H}_\infty$ controllers that is based on this parametrization see Péni et al. [2014].