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Bicoprime Factor Stability Criteria and Uncertainty Characterisation ⋆

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Abstract: Bicoprime factorisations can be considered a generalisation of the well known coprime factorisations. This paper deals with the possible applications of such factorisations in control theory. The concept of minimal dimension bicoprime factorisations is introduced and shown to be especially beneficial in the case of normal rank deficient and redundant control systems. Some methods of obtaining a bicoprime factorisation for a plant are given based on state-space data, right or left coprime factorisations and left or right standard factorisations. Finally, bicoprime factor uncertainty is characterised and shown to have an interesting structure closely resembling the standard four-block uncertainty.

Keywords: Factorization methods, coprime factorizations, stability criteria, redundant control, uncertainty

1. INTRODUCTION

Coprime factorisations over $\mathscr{RH}_\infty$ have been extensively studied by the control community. They have many applications in control theory, forming the basis of many important results. For example, distance measures such as the gap, graph and ν-gap metrics can be posed as normalised coprime factor model matching problems as demonstrated in Georgiou and Smith (1990); Vidyasagar (1984) and Vinnicombe (2001). Additionally, solutions to the $\mathscr{H}_\infty$ loopshaping robust stabilisation problem use normalised coprime factorisations of the plant (Glover and McFarlane, 1989). Coprime factorisations have also been used to validate controllers for internal stability and robust performance using closed loop data (Dehghani et al., 2009; Patra and Lanzon, 2012).

Bicoprime factorisations (BCFs) were first introduced by Vidyasagar (2011) [a] as a generalisation of standard left or right coprime factorisations. This fact will be exemplified with the derivation of bicoprime (BC) factor stability results in Section 4. Briefly studied in the late 1980's, some results were derived including stability of the feedback interconnection of a plant given as a BC treble and a controller expressed as either a left coprime factorisation (LCF) or right coprime factorisation (RCF) (Desoer and Gündeş, 1988; Gündeş and Desoer, 1990). This was achieved by transforming the given BCF into a RCF or LCF and using existing results. However, those early results are far from a comprehensive study of the subject matter.

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[a] This is a reprint of the 1985 original published by M.I.T. Press.

Two motivating points for the study of BCFs given in Vidyasagar (2011) are as follows. First, they naturally emerge in closed loop transfer matrices. Indeed, most proofs in this paper use this fact to establish their results. Second, any minimal state-space realisation of a plant is a BCF over $\mathscr{R}$. This follows directly from the well known Popov-Belevitch-Haututs tests. In fact, a number of factorisations can be related to BCFs over different spaces. Other examples include spectral and Wiener-Hopf factorisations which can be considered as BCFs over $\mathscr{RL}_\infty$ with some extra impositions on the factors. Further to these points, it can also be argued that BCFs, being a generalised version of RCFs and LCFs, provide a link between the two, unifying any results. The duality of LCFs and RCFs has been well established for some time now, with BCFs demonstrating how this duality arises.

BCFs received some attention in the study of decentralized control. Ünyelioğlu et al. (2000) shows that BC factors can be used to characterise fixed zeros in decentralized control and by extent deduce the existence of a decentralized stabilising controller for a given plant. Additionally, Baski et al. (1999) presents methods for the decentralized stabilisation of plants using BCFs.

Another possible application for BCFs that emerges from the results of this paper is in the area of redundant control systems. Using more actuators or sensors than needed for the purposes of fault tolerance can lead to a rank deficient system. The use of BCFs can be beneficial to the study of such systems. Using a BCF stability test leads to reduced dimension stability matrices. It will be shown by example that in some instances the stability of a closed loop transfer matrix can be established by the invertibility in $\mathscr{RH}_\infty$ of a single SISO transfer function or equivalently if it has any right half plane zeros.

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There is a considerable amount of mathematical richness associated with BCFs that so far remains untapped by the control community. Moreover, most results pertaining to BCFs exhibit a considerable amount of mathematical symmetry which makes them a very appealing subject of study. As mentioned above BCFs are a generalised version of coprime factorisations. This leads to more complex mathematical results than in the classical case. However, this complexity should not serve as a deterrent as the potential advancements to control theory necessitate further investigation.

2. PRELIMINARIES

This section defines the notation that will be used throughout this paper and recalls some standard, well-known results, mostly related to stability and coprime factorisations.

2.1 Notation

$\mathscr{R}$	Set of proper real-rational transfer matrices
$\mathscr{RH}_\infty$	Set of proper real-rational stable transfer matrices
$\mathscr{GH}_\infty$	$\{Q \in \mathscr{RH}_\infty : Q^{-1} \in \mathscr{RH}_\infty\}$
$\mathrm{r}\,(A)$	Rank of the matrix $A \in \mathbb{C}^{m\times n}$
$\mathrm{nr}\,(P)$	Normal rank of the transfer matrix $P \in \mathscr{R}$
$\mathrm{diag}\,(\cdot)$	Block diagonal matrix starting from the top left
$\mathrm{adiag}\,(\cdot)$	Block anti-diagonal matrix starting from the top right
$A^\dagger$	Pseudo-inverse of $A \in \mathbb{C}^{m\times n}$
A^*	Complex conjugate transpose of $A \in \mathbb{C}^{m\times n}$
$\bar{s}$	Complex conjugate of $s \in \mathbb{C}$
$\mathbb{C}_{<0}$	$\{s : s \in \mathbb{C}, \Re(s) < 0\}$
$\mathbb{C}_{\le 0}$	$\{s : s \in \mathbb{C}, \Re(s) \le 0\}$
$\mathbb{C}_{>0}$	$\{s : s \in \mathbb{C}, \Re(s) > 0\}$
$\mathbb{C}_{\ge 0}$	$\{s : s \in \mathbb{C}, \Re(s) \ge 0\}$
$\mathcal{F}_l(H,\Delta)$	Lower LFT of H with respect to Δ
$\mathcal{F}_u(H,\Delta)$	Upper LFT of H with respect to Δ

2.2 Stability

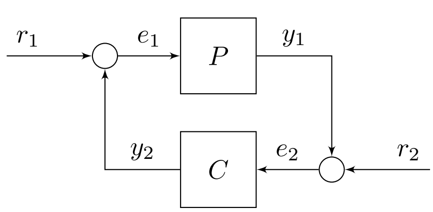

Fig. 1. Standard feedback interconnection of P and C.

The feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$, shown in Figure 1 and denoted by $[P, C]$, is well posed if all transfer matrices from (r_1, r_2) to (e_1, e_2) are well-defined and proper. A necessary and sufficient condition for this to be true is $\det(I - PC)(\infty) \neq 0$ (Zhou et al., 1996, Lemma 5.1). The transfer matrix from $(-r_2, r_1)$ to (y_1, e_1) is denoted by $H(P, C)$.

Lemma 1. (Zhou et al. (1996) Lemma 5.3). The feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ is internally stable if and only if

$$\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} \in \mathscr{RH}_\infty$$

or equivalently

$$H(P, C) = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \;\; I] \in \mathscr{RH}_\infty.$$

Lemma 2. Let $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right] \in \mathscr{RH}_\infty$ and suppose that $A_{22} \in \mathscr{GH}_\infty$. Then

$$A \in \mathscr{GH}_\infty \Leftrightarrow A_{11} - A_{12}A_{22}^{-1}A_{21} \in \mathscr{GH}_\infty.$$

Proof. From the Schur complement decomposition

$$A = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}.$$

Since by supposition $A_{22} \in \mathscr{GH}_\infty$,

$$\begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix} \in \mathscr{GH}_\infty$$

and the result follows. □

2.3 Coprime factorisations

Definition 1. (Zhou et al. (1996) Definition 5.3). The pair $\{N, M\}$ is right coprime (RC) in $\mathscr{RH}_\infty$ if $N, M \in \mathscr{RH}_\infty$ and there exist $Y_r, Z_r \in \mathscr{RH}_\infty$ such that $Z_rM + Y_rN = I$. Furthermore, the pair is a RCF of a plant P over $\mathscr{RH}_\infty$ if M is square, $\det M(\infty) \neq 0$ and $P = NM^{-1}$.

Definition 2. (Zhou et al. (1996) Definition 5.3). The pair $\{L, M\}$ is left coprime (LC) in $\mathscr{RH}_\infty$ if $M, L \in \mathscr{RH}_\infty$ and there exist $Y_l, Z_l \in \mathscr{RH}_\infty$ such that $MZ_l + LY_l = I$. Furthermore, the pair is a LCF of a plant P over $\mathscr{RH}_\infty$ if M is square, $\det M(\infty) \neq 0$ and $P = M^{-1}L$.

Definition 3. The set of all RC (resp. LC) pairs in $\mathscr{RH}_\infty$ is defined as $\mathscr{C}_r$ (resp. $\mathscr{C}_l$). Similarly, the set of all RCFs (resp. LCFs) of a plant P over $\mathscr{RH}_\infty$ is defined as $\mathscr{C}_r(P)$ (resp. $\mathscr{C}_l(P)$).

Definition 4. Let $\{N, M\} \in \mathscr{C}_r$ and $\{L, M\} \in \mathscr{C}_l$. The associated Bézout factor sets are defined as [b]

$$\mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] = \{\{Y_r, Z_r\} : Y_r, Z_r \in \mathscr{RH}_\infty, Z_rM + Y_rN = I\},$$
$$\mathscr{C}^\dagger \left[\begin{smallmatrix} M & L \end{smallmatrix}\right] = \{\{Y_l, Z_l\} : Y_l, Z_l \in \mathscr{RH}_\infty, MZ_l + LY_l = I\}.$$

Some well known coprime factor stability results are listed in following lemma.

Lemma 3. (Zhou et al. (1996) Lemma 5.2). Let $\{N, M\} \in \mathscr{C}_r(P)$, $\{L, \tilde{M}\} \in \mathscr{C}_l(P)$, $\{U, V\} \in \mathscr{C}_r(C)$ and $\{W, \tilde{V}\} \in \mathscr{C}_l(C)$. Then the following statements are equivalent.

(1) $[P, C]$ is internally stable
(2) $\tilde{M}V - LU \in \mathscr{GH}_\infty$
(3) $\tilde{V}M - WN \in \mathscr{GH}_\infty$
(4) $\begin{bmatrix} M & U \\ N & V \end{bmatrix} \in \mathscr{GH}_\infty$
(5) $\begin{bmatrix} \tilde{M} & -L \\ -W & \tilde{V} \end{bmatrix} \in \mathscr{GH}_\infty$

3. BICOPRIME FACTORISATIONS

In their original definition, BCFs of a plant were presented as a quad of objects in $\mathscr{RH}_\infty$.

[b] The use of the pseudo-inverse symbol (†) is appropriate in the above definition since $[Z_r \; Y_r]$ is the left inverse of $[M^* \; N^*]^*$ and $[Z_l^* \; Y_l^*]^*$ is the right inverse of $[M \; L]$.