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Robust control design based on convex liftings

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Abstract: In control related studies, convex liftings have been of use to solve inverse parametric linear/quadratic programming problem. This paper presents a so-called *convex liftings based method* for robust control design of constrained linear systems affected by bounded additive disturbances. It will be shown that a geometrical construction as convex lifting can be used in optimization-based control design to guarantee robust stability and recursive feasibility in a given controllable region of the state space. Finally, a numerical example will be considered to illustrate this method.

1. INTRODUCTION

Robust control plays an important role in control theory. In particular, for constrained discrete-time linear systems, robust control design in the presence of bounded additive disturbances and/or polytopic uncertainty, has been of interest in countless studies. Different design techniques have been put forward as in Kothare et al. [1996], Scokaert and Mayne [1998], Mayne et al. [2005], Rakovic et al. [2012], Bemporad et al. [2003], Grancharova and Johansen [2012], Gutman and Cwikel [1987], Blanchini [1994, 1995], Nguyen [2014], etc.

Linear matrix inequality (LMI) has been early applied in model predictive control (MPC), in Kothare et al. [1996] to design robust controller in the presence of polytopic model uncertainties. This method requires at each sampling time solving a computationally demanding LMI problem. Subsequently, based on dynamic programming, min-max optimization based method in Scokaert and Mayne [1998] has solved an MPC problem for discrete-time, linear invariant systems subject to bounded, additive disturbances. This method aims to minimize at each sampling instant, the worst case of cost function, subject to exponentially increasing set of constraints once the prediction horizon increases. Later, it is shown that a robust linear MPC problem can be alternatively solved via parametric convex programming to design explicit robust control in the presence of bounded additive disturbances and polytopic model uncertainties, see e.g. Bemporad et al. [2003]. Also, approximation of robust explicit control laws for nonlinear MPC in the presence disturbances has been studied in Grancharova and Johansen [2012]. On the other hand, tube based MPC has been originated in Mayne et al. [2005] and developed in Rakovic et al. [2012], providing new insight in robust control design. Another line of robust control was originated in Gutman and Cwikel [1987] based on *positively invariant sets* and has bloomed via different studies e.g. Blanchini [1994], Nguyen [2014] showing their simple formulations and easy implementations.

In the same line with the last studies, this paper introduces another approach based on *convex liftings* which can serve as Lyapunov functions. This method will be proved to guarantee the recursive feasibility and closed loop stability. In terms of implementation, it only requires solving a simple linear programming problem at each sampling instant.

Convex liftings have been used in studies related to structural properties of parametric convex programming based control laws. To our best knowledge, the present approach is the first attempt to use convex lifting as a direct design method.

Notation

Throughout this paper, $\mathbb{N}, \mathbb{N}_{>0}, \mathbb{R}, \mathbb{R}_+$ denote the set of non-negative integers, the set of strictly positive integers, the set of real numbers and the set of non-negative real numbers, respectively. For ease of presentation, with a given $N \in \mathbb{N}_{>0}$, by $\mathcal{I}_N$, we denote the index set: $\mathcal{I}_N = \{i \in \mathbb{N}_{>0} \mid i \leq N\}$.

A polyhedron is the intersection of finitely many halfspaces. A polytope is a bounded polyhedron. If P is an arbitrary polytope, then by $\mathcal{V}(P)$, we denote the set of its vertices. If $\mathcal{S}$ is a finite set, then $\mathrm{conv}(\mathcal{S})$ denotes the convex hull of $\mathcal{S}$. Also, for a given set $\mathcal{S}$, by $\mathrm{int}(\mathcal{S})$, we denote the interior of $\mathcal{S}$. Further, we use $\dim(\mathcal{S})$ to denote the dimension of its affine hull.

Given a set $\mathcal{S} \subset \mathbb{R}^d$ and a matrix $A \in \mathbb{R}^{d\times d}$, then $A\mathcal{S}$ is defined as follows: $A\mathcal{S} = \{As \mid s \in \mathcal{S}\}$.

Given two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^d$, their Minkowski sum is denoted by $\mathcal{S}_1 \oplus \mathcal{S}_2$ and is defined by:

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \left\{x \in \mathbb{R}^d \mid \exists y_1 \in \mathcal{S}_1, y_2 \in \mathcal{S}_2 \text{ s.t. } x = y_1 + y_2\right\}.$$

Also, $\mathcal{S}_1 \backslash \mathcal{S}_2$ is defined as follows:

$$\mathcal{S}_1 \backslash \mathcal{S}_2 := \left\{x \in \mathbb{R}^d \mid x \in \mathcal{S}_1,\ x \notin \mathcal{S}_2\right\}.$$

Further, given two different points $x, y \in \mathbb{R}^d$, we use $\rho(x, y)$ to denote the Euclidean distance between x and y. If $y = 0$, this distance is briefly written by $|x|$. Moreover, given a set $\mathcal{A} \subset \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, we denote $\rho_{\mathcal{A}}(x) = \inf_{y\in\mathcal{A}} \rho(x, y)$. It is clear that if $x \in \mathcal{A}$, then $\rho_{\mathcal{A}}(x) = 0$. The distance from a point to a set is also known as the Hausdorff distance and can be understood as a particular case of the distance between two sets.

This paper is organized in five sections. The problem statement is presented in Section 2. Our main results will be introduced in Section 3. An illustrative example will be considered in Section 4. The final section summarizes the contribution of the present paper.

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2. PROBLEM SETTING

In this paper, we concentrate on the class of discrete-time linear invariant systems, affected by bounded additive disturbances:

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad (1)$$

where x_k, u_k denote the state, control variables at time k, w_k stands for the disturbance at time k. The state, control variables and the disturbances are subject to constraints:

$$x_k \in \mathbb{X} \subset \mathbb{R}^{d_x}, u_k \in \mathbb{U} \subset \mathbb{R}^{d_u}, w_k \in \mathbb{W} \subset \mathbb{R}^{d_x}, \quad (2)$$

where $d_x, d_u \in \mathbb{N}_{>0}$, $\mathbb{X}, \mathbb{U}, \mathbb{W}$ are polytopes. It is assumed that $\mathbb{X}, \mathbb{U}, \mathbb{W}$ contain the origin in their interior.

The aim is to find a state feedback control law which exhibits robustness with respect to additive disturbances such that the closed loop is robustly stable. It is clear that if disturbance w_k is unknown for the computation of control action at instant k, one cannot expect to be able to guarantee the asymptotic stability of the origin. The asymptotic stability is replaced with an ultimate boundedness notion Khalil [2002], Kofman et al. [2007]. The following classical assumption is necessary for the existence of stabilizing control laws.

Assumption 2.1. The pair (A, B) is stabilizable and full-state measurement is available for control.

3. CONVEX LIFTINGS BASED CONTROL DESIGN

3.1 Disturbance invariant sets with respect to a stabilizing control law

Positively invariant sets have been studied over three decades. Due to their relevance in control theory, they turn out to be of help in many control related studies e.g. Bitsoris [1988], Blanchini and Miani [2007], Rakovic et al. [2012], Nguyen [2014],

In particular, disturbance invariant sets are meaningful in robust control design for system (1). Some remarkable results on the structure, properties and algorithms for positively invariant sets can be found in Kolmanovsky and Gilbert [1998], Rakovic et al. [2005, 2004].

The definition of a positively invariant set for linear system (1) is recalled below.

Definition 3.1. Given the dynamic system (1) subject to constraints (2), with respect to Assumption 2.1, a set Ω is called *positively invariant* with a linear control law $u_k = Kx_k \in \mathbb{U}$ if and only if $(A + BK)\Omega \oplus \mathbb{W} \subseteq \Omega$.

Such an Ω defined in Definition 3.1 is alternatively called *disturbance invariant set*. Algorithms for approximating maximal and minimal disturbance invariant sets can be found in Kolmanovsky and Gilbert [1995, 1998], Rakovic et al. [2005], Gilbert and Tan [1991]. It will be considered in the developments, presented next, that such approximations are available for control design.

Also, for the linear system (1) satisfying Assumption 2.1, it is easy to find a linear stabilizing state feedback $u_k = Kx_k$ via the solution of the Riccati equation with a pre-chosen, positive semidefinite weighting matrices, Q, R. The influence of disturbances can be taken into account in the design of unconstrained stabilizing linear feedback Boyd et al. [1994].

Note that in the presence of persistent disturbances, Ω is considered as a full-dimensional set. Otherwise, if system (1) is not affected by additive disturbances and/or is subject to polytopic model uncertainties, $\Omega = \{0\}$ can also be chosen. However, these cases are beyond the scope of this paper.

3.2 Domain of attraction

A domain of attraction is known to be a subset of all points which can be driven to a target set. To guarantee the convergence to a disturbance invariant set Ω, a domain of attraction denoted by $\mathcal{X}$, should ensure that for any point belonging to $\mathcal{X}$, there always exists control law satisfying constraint (2), which steers the state to Ω. The following definition of a contractive set, inherited from Definition 2.5 in Blanchini [1994], is of help for our development.

Definition 3.2. Given λ, $0 \leq \lambda \leq 1$, a set S is called $\lambda-$*contractive* if for any $x \in S \subseteq \mathbb{X}$, there exists $u(x) \in \mathbb{U}$ such that $(Ax + Bu(x)) \oplus \mathbb{W} \subseteq \lambda S$. If $\lambda = 1$, S is said *control invariant*.

According to Blanchini [1994], the maximal $\lambda-$contractive set, denoted by P_λ, is defined as the set containing all $\lambda-$contractive sets for system (1) subject to constraint (2). A computation of this set is recalled as follows.

$$\begin{aligned} &S_1 = \mathbb{X}, \\ &S_{i+1} = \{x \in \mathbb{X} \mid \exists u(x) \in \mathbb{U}, \text{ s.t.} \\ &\qquad\qquad (Ax + Bu(x)) \oplus \mathbb{W} \subseteq \lambda S_i\}, \text{ for } i \in \mathbb{N}_{>0}, \\ &P_\lambda = S_\infty. \end{aligned}$$

Details about algorithms for computation of P_λ can be found in Blanchini [1994], Kerrigan [2001]. For our development, we will use the maximal $\lambda-$contractive set P_λ for $0 \leq \lambda < 1$, as a domain of attraction; i.e. $\mathcal{X} = P_\lambda$.

3.3 Convex lifting construction

Convex lifting is in principle a purely geometrical notion. In control theory, the optimal cost function to a parametric linear programming problem, known as a convex lifting, is used to facilitate the implementation of explicit control laws, see e.g. Baotic et al. [2008], Jones et al. [2006]. Subsequently, it has been of use to solve inverse parametric linear/quadratic programming problem in Nguyen et al. [2014b,a, 2015]. It is worth stressing that the term "convex function" deployed in Hempel et al. [2013, 2015] completely differs from a convex lifting defined here. Before recalling its definition, some additional notation will be introduced.

Definition 3.3. A collection of $N \in \mathbb{N}_{>0}$ full-dimensional polyhedra denoted as $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$, is called a *polyhedral partition of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ if:

- $\bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i = \mathcal{X}$.
- $\text{int}(\mathcal{X}_i) \bigcap \text{int}(\mathcal{X}_j) = \emptyset$ with $i \neq j$, $(i, j) \in \mathcal{I}_N^2$,

Also, $(\mathcal{X}_i, \mathcal{X}_j)$ are called neighbours if $(i, j) \in \mathcal{I}_N^2$, $i \neq j$ and $\dim(\mathcal{X}_i \cap \mathcal{X}_j) = d_x - 1$. Also, if $\mathcal{X}$ is a polytope, then $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ is called *a polytopic partition*.

Definition 3.4. For a given polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^{d_x}$, a *piecewise affine lifting* is described by the function $z : \mathcal{X} \to \mathbb{R}$ with:

$$z(x) = a_i^T x + b_i \text{ for any } x \in \mathcal{X}_i, \quad (3)$$

and $a_i \in \mathbb{R}^{d_x}$, $b_i \in \mathbb{R}$, $\forall i \in \mathcal{I}_N$.