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Polynomial Stabilization with Bounds on the Controller Coefficients ⋆

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Abstract: Let b/a be a strictly proper reduced rational transfer function, with a monic. Consider the problem of designing a controller y/x, with $\deg(y) \le \deg(x) < \deg(a) - 1$ and x monic, subject to lower and upper bounds on the coefficients of y and x, so that the poles of the closed loop transfer function, that is the roots (zeros) of $ax + by$, are, if possible, strictly inside the unit disk. One way to formulate this design problem is as the following optimization problem: minimize the root radius of $ax + by$, that is the largest of the moduli of the roots of $ax + by$, subject to lower and upper bounds on the coefficients of x and y, as the stabilization problem is solvable if and only if the optimal root radius subject to these constraints is less than one. The root radius of a polynomial is a non-convex, non-locally-Lipschitz function of its coefficients, but we show that the following remarkable property holds: there always exists an optimal controller y/x minimizing the root radius of $ax + by$ subject to given bounds on the coefficients of x and y with *root activity* (the number of roots of $ax + by$ whose modulus equals its radius) and *bound activity* (the number of coefficients of x and y that are on their lower or upper bound) summing to at least $2\deg(x) + 2$. We illustrate our results on two examples from the feedback control literature.

Keywords: Robust control, polynomial optimization, frequency domain stabilization

1. INTRODUCTION

Let $\mathcal{P}_n$ denote the space of real polynomials of degree n or less and let $\mathcal{P}_n^1$ denote the monic real polynomials of degree n. Let a rational function b/a be given, with $\deg(b) < \deg(a)$ and with a monic. We wish to design a controller y/x for b/a, with $\deg(y) \le \deg(x) < \deg(a) - 1$ and x monic, so that the poles of the closed loop transfer function, equivalently the roots (zeros) of $ax + by$, all lie inside the unit disk, subject to prescribed lower and upper bounds on the coefficients of x and y. Define the *root radius* ρ of a polynomial $p \in \mathcal{P}_n^1$ as

$$\rho(p) = \max\{|\lambda| \ : \ p(\lambda) = 0\},$$

the maximum of the moduli of its roots. Clearly, b/a can be stabilized by y/x with the required constraints if and only if the global minimum of the root radius $\rho(ax + by)$ subject to the required constraints on x and y is less than one. The root radius is a non-convex function and it is not locally Lipschitz at polynomials with multiple roots. Nonetheless, it has a remarkable property that we explain in the next section.

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2. A THEOREM ON ROOT AND BOUND ACTIVITY

Let coeff : $\mathcal{P}_n^1 \to \mathbb{R}^n$ be defined by

$$\text{coeff}(z^n + c_{n-1}z^{n-1} + \cdots + c_0) = [c_0, c_1, \ldots, c_{n-1}]^T.$$

Theorem 1. Let a and b be fixed polynomials with no non-constant common factors, with $\deg(b) < \deg(a)$, and with a monic. Let $0 \le d \le \deg(a) - 2$ and consider the optimization problem:

$$\min_{x \in \mathcal{P}_d^1, y \in \mathcal{P}_d} \{\rho(ax + by) : \ell \le \text{coeff}(x) \le u, \ell \le \text{coeff}(y) \le u\}$$

where $\ell \in \mathbb{R} \cup \{-\infty\}$, $u \in \mathbb{R} \cup \{\infty\}$, $\ell < u$ and the inequalities are to be interpreted componentwise. Then there always exists a globally optimal minimizer $ax^* + by^*$ for which the *root activity* of $ax^* + by^*$ (the number of its roots, counting multiplicity, whose modulus equals $\rho(ax^* + by^*)$) and the *bound activity* (the number of coefficients of x^* and y^* that are on their lower or upper bound) sum to at least $2d + 2$.

Sketch of proof. Let n denote the degree of $ax + by$, so $n = \deg(a) + d$, and let m denote the number of free variables in x and y, so $m = 2d + 1$. The argument that follows requires that, when no bounds are active, the resulting number of implicit affine equality constraints on $\mathcal{P}_n^1$, say k, is exactly $n - m = \deg(a) - d - 1$. For this

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to be true, the map $(x, y) \mapsto (ax + by)$ needs to have the property that it is one-to-one, that is, $ax + by = ax' + by'$ implies that $x = x'$ and $y = y'$. Since the map is linear, this is equivalent to: $ax+by = 0$ only if $x = y = 0$. Suppose that $ax + by = 0$ but $y \neq 0$. (If $y = 0$, the only solution to $ax = 0$ is $x = 0$, which is impossible since x is monic.) Then $ax = -by$, which implies that $x/y = -b/a$ (note that both y and a are nonzero). In order for the rational functions x/y and $-b/a$ to be equal, the latter must be reducible since $\deg(a) > \deg(y)$, contradicting the assumption on a and b. Therefore the mapping $(x, y) \mapsto ax + by$ is one-to-one and we can conclude that we have exactly $\deg(a)-d-1$ implicit affine equality constraints.

Our argument proceeds as in the proof of (Eaton et al., 2014, Theorem 4.1), which establishes the result when $\ell = -\infty$, $u = \infty$. Let p_* be an optimal solution, which exists because the lower level sets of ρ are bounded, and let n_B be the number of active bounds for p_*. If $n_B > 0$, freeze the corresponding variables. This decreases the number of free variables m from $2d+1$ to $2d+1-n_B$ or, equivalently, increases the number of implicit affine equality constraints k on $\mathcal{P}_n^1$ from $\deg(a) - d - 1$ to $n_B + \deg(a) - d - 1$.

Let n_A be the number of active roots of p_*, that is, the number of its roots, counting multiplicity, whose modulus equals $\rho(p_*)$. If $n_A \geq m+1 = 2d+2-n_B$, there is nothing more to show. So suppose $n_A \leq 2d + 1 - n_B$.

Factor p_*, which is a monic polynomial of degree $n = \deg(a) + d$, as $p_* = q^A q^I$, where q^A and q^I are both monic, such that the roots of q^A are the active roots of p_* and the roots of q^I are the inactive roots of p_*. We have $\deg(q^A) = n_A$ and $\deg(q^I) = n - n_A$. By assumption, $\deg(q^I) \geq \deg(a) + n_B - d - 1 = k$.

We now construct an affine perturbation q_t^I of q^I, with $t \in \mathbb{R}$ and $q_0^I = q^I$. It is shown in Eaton et al. (2014) that because $\deg(q^I) \geq k$, such a perturbation can be made by suitable changes to the variables, or equivalently, remaining feasible with respect to the implicit affine equality constraints. If $\deg(q^I) = k$, normally it is necessary to perturb all k variable coefficients of q^I (those corresponding to $1, z, \cdots, z^{k-1}$). If $\deg(q^I) > k$, we may restrict the perturbation to any k of these coefficients. Note that only inactive roots of $q^A q_t^I$ depend on t, and if t is increased from zero by a sufficiently small amount, they remain inactive, but if we increase t enough, say to a critical value t_*, we will either arrive at a new polynomial with an additional active root (or roots), increasing n_A, or we will hit a variable bound (or bounds), increasing n_B, or both. If the resulting increased value of $n_A + n_B$ is at least $2d + 2$, there is nothing more to show. Otherwise, we return to the factorization step and repeat the argument, factoring the new polynomial $q^A q_{t_*}^I$ in the same way. For example, suppose that a single new active real root was encountered, so n_A increased by one: then we move the associated linear factor of $q_{t_*}^I$ into q^A and the remaining part of $q_{t_*}^I$ becomes the new q^I. If a new active complex conjugate pair of roots was encountered, so n_A increased by two, we move the associated quadratic factor of $q_{t_*}^I$ into q^A and the remaining part of $q_{t_*}^I$ becomes the new q^I. If we encountered a new bound, so n_B increased, we freeze the corresponding variable, reducing m, the number of free variables, by one, and simply set the new q^I to $q_{t_*}^I$. Since each step results in either an increase in the number of active roots or an increase in the number of active bounds, their sum will eventually reach $2d + 2$.

3. EXAMPLES

We illustrate Theorem 1 using two examples from the literature.

Example 1. (Bhattacharyya et al., 2009, p.167). In this example, $a(z)$ and $b(z)$ are respectively given by

$$z^5 - 0.2z^4 - 3.005z^3 - 3.9608z^2 - 0.0985z + 1.2311$$

and

$$z^4 + 1.93z^3 + 2.2692z^2 + 0.1443z - 0.7047.$$

The open-loop system is unstable as $\rho(a) = 2.2629$.

Example 2. Tong and Sinha (1994); Henrion et al. (2003). For this robot example, after a suitable change of notation to translate the example into our setting, we have that $a(z)$ and $b(z)$ are respectively

$$z^8 - 2.914z^7 + 3.6930z^6 - 2.8055z^5 + 1.2773z^4 - 0.2508z^3$$

and

$$0.0257z^3 - 0.0764z^2 - 0.1619z - 0.1688.$$

The open-loop system is marginally unstable as $\rho(a) = 1$.

We do not have a method to find global minimizers of max root optimization problems regardless of whether bounds are present, so we approximated them using a local optimization method run from many starting points. As explained by Lewis and Overton (2013), the BFGS quasi-Newton method, which was originally developed to minimize differentiable functions, is also extremely effective for finding local minimizers of non-smooth functions. It can even be applied to non-locally-Lipschitz functions such as the root radius, although accurate results cannot be expected when the root radius is not Lipschitz at a computed minimizer. To account for the bound constraints, we minimized the penalty function $P(x, y) = \rho(ax + by) + wv(x, y)$, where $v(x, y)$ is the L_1 norm of the bound violations and the penalty parameter w is increased as needed to obtain feasible solutions (Fletcher (2000)). To search for minimizers of $P(x, y)$, we ran BFGS from 100 randomly generated starting points for each problem instance, with the hope that for small problem instances, global minima will be found. We ran experiments with d, the degree of x and y, ranging from 0 to $\deg(a) - 2$. In each case we made one set of runs without bounds (i.e., $\ell = -\infty$, $u = \infty$), and two other sets of runs with finite bounds imposed. Naturally, the tighter the bounds, the more difficult it is to stabilize the system, but the optimization problems become easier to solve accurately as high-multiplicity roots are less likely to occur. For each d and choice of bounds, let $a\tilde{x} + b\tilde{y}$ denote the computed optimal polynomial, that is, the $ax + by$ with the lowest root radius found by BFGS over the 100 starting points, subject to the bounds that were imposed on x and y.

Tables 1 through 6 show, for each d, the degree $n = \deg(a)+d$ of $ax+by$, the estimated root activity for $a\tilde{x}+b\tilde{y}$ (the number of its roots whose modulus equals $\rho(a\tilde{x} + b\tilde{y})$, within a small tolerance), the bound activity (the number of coefficients of x and y on their lower or upper bounds), the sum of the two activities, and, for comparison, the