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Stabilization of Affine Systems with Control Singularities ⋆

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Abstract: This paper deals with the zero equilibrium stabilization problem for affine systems that have control input singularities. We consider a class of scalar input systems written in a canonical form with the input coefficient vanishing to zero on a set of points in the state space that includes the origin. The necessary and sufficient conditions are obtained for the zero equilibrium stabilizability of such affine systems. As an example, stabilization of third-order dynamical systems by means of constant and variable state feedback controls is considered.

Keywords: Nonlinear control, Stabilization, Affine systems, Singularities

1. INTRODUCTION

One of the serious drawbacks to stabilizing a nonlinear dynamical system is the presence of control input singularities, i.e. existence of a set of points in the state space where the coefficient of input becomes null. This restricts the applicability of such well known nonlinear control techniques as feedback linearization, integrator backstepping and sliding mode control. Even standard Jacobian linearization may fail since the linear approximation of such a system about a singular point is independent of the control input.

The control singularity problem has motivated much research during the past decades, see e.g. Chen and Ballance (2002), Commuri and Lewis (1995), Emel'yanov and Krishchenko (2012), Guo et al. (1996), Hauser et al. (1992), Leith and Leithead (2001), Li and Krstić (1997), Ratliff and Pagilla (2006), Sun et al. (2009), Tomlin and Sastry (1998), Yeom and Choi (2006), Zhang et al (2008). In case of input-output linearization, the input singularity appears for the non-regular systems, i.e. those that do not have a well-defined relative degree (see e.g. Isidori (1995)). For the class of strict feedback systems, the virtual control singularities turn out to be an essential obstacle to applying the integrator backstepping technique, see Li and Krstić (1997).

One of the early ideas to cope with the input-output linearizing control singularities was to use an approximate feedback linearization technique which is based on approximating the system in question by another nonlinear system that is feedback linearizable, see Hauser et al. (1992). This is done by neglecting some higher order terms that lead to the singularity. In Zhang et al (2008) it is shown that neighborhood of the input-output linearizing control singularity can be divided into two regions. In one of the regions, the system still may have a well-defined relative degree and is input-output linearizable. The work of Leith and Leithead (2001) provides sufficient conditions for existence of arbitrarily accurate approximate input-output linearizing control laws. However, approximate feedback linearization allows to achieve only local stabilization results that are valid close to the singular points, if applicable.

A more promising approach is to use the feedback linearization control away from the singularities and switch to some other control law, for instance, an approximate feedback linearization control, when the state is close to the singular points, see Chen and Ballance (2002), Tomlin and Sastry (1998), Zhang et al (2008). Such technique helps avoid singularities of the control law and, meanwhile, broaden the domain of applicability of approximate stabilizing feedbacks. More control switching schemes can be found, for instance, in Commuri and Lewis (1995), Emel'yanov and Krishchenko (2012), Guo et al. (1996),Yeom and Choi (2006).

However, at present, there is no general answer to the question if stabilization of a nonlinear system is possible in the presence of control input singularities. In this paper, we are interested in the problem of stabilizability of smooth dynamical systems that are given in the following canonical form

$$y^{(n)} + f(y, \dot{y}, \ldots, y^{(n-1)}) = g(y, \dot{y}, \ldots, y^{(n-1)})u, \quad (1)$$

where $\overline{y} = (y, \dot{y}, \ldots, y^{(n-1)}) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, $f(0) = 0$, $g(\overline{y})$ is equal to zero on a set of points in the state space, that contains the origin.

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Hence, the origin $\overline{y} = 0$ is an equilibrium point of the system (1).

Notice that the conventional feedback linearization control law fails to stabilize the zero equilibrium of the system (1), since the function $1/g(\overline{y})$ becomes unbounded on the set $\{\overline{y}\,|\,g(\overline{y}) = 0\}$ of singular points and is not defined on this set.

The key results provided by this note are the necessary conditions for stabilizability of nonlinear systems of the form (1). In the third-order case, we find sufficient conditions for stabilizability of the origin by means of constant controls. This paper also suggests a feedback control switching scheme to practically stabilize (see e.g. Byrnes (2008)) the origin in spite of the presence of control input singularities.

The remaining of the paper is organized as follows. The necessary conditions for stabilizability of affine systems given by (1) are obtained in section 2. Section 3 contains the sufficient conditions for stabilizability of third-order affine systems by constant controls. In section 4 the suggested feedback control switching scheme is discussed. An illustrative example and numerical simulation results are presented in section 4 to illustrate the performance of the proposed controller. Finally, the paper concludes with some remarks in Section 5.

Notation: Further in the paper, ∂S denotes the boundary of a set $S \subset \mathbb{R}^n$. For any two vectors $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ the inequality $y \geq z$ implies that $y_i \geq z_i$, $i = \overline{1,n}$. For a vector $x \in \mathbb{R}^n$, $\|\,x\,\|$ stands for the Euclidean norm.

2. NECESSARY CONDITIONS FOR STABILIZABILITY

Consider a smooth affine system of the form (1) written as

$$y^{(n)} + f(\overline{y}) = (y^{(n-1)} - \varphi(\hat{y}))u, \tag{2}$$

where $\hat{y} = (y, \dot{y}, \dots, y^{(n-2)})$, $\varphi(0) = 0$. Let $a_j = f'_{y^{(j)}}(0)$, $j = \overline{0, n-1}$; $l_i = \varphi'_{y^{(i)}}(0)$, $i = \overline{0, n-2}$. We also define $\hat{l} = (l_0, l_1, \dots, l_{n-2})$ and $\hat{\xi} = (\xi_0, \xi_1, \dots, \xi_{n-2})$, where

$$\xi_0 = a_{n-1}l_0 + a_0 + l_{n-2}l_0,$$
$$\xi_i = a_{n-1}l_i + a_i + l_{n-2}l_i + l_{i-1},\ i = \overline{1, n-2}.$$

Theorem 1. If $\hat{l} > 0$; $\hat{\xi} < 0$, then there is no control law that stabilizes the equilibrium point $\overline{y} = 0$ of the system (2).

Proof. Let

$$g(\overline{y}) = y^{(n-1)} - \varphi(\hat{y}).$$

To examine orientation of the vector field

$$\mathbf{F} = \Big\{\dot{y},\ \dots,\ y^{(n-1)},\ -f(\overline{y}) + g(\overline{y})u\Big\}$$

of the system (2) on the surface $g(\overline{y}) = 0$, compute the derivative

$$\mathbf{F}g(\overline{y}) = \frac{dg}{dt}\Big|_{(2)} = \big(y^{(n)} - \sum_{i=0}^{n-2} \varphi'_{y^{(i)}}(\hat{y})y^{(i+1)}\big)\Big|_{(2)}$$
$$= -f(\overline{y}) + g(\overline{y})u - \sum_{i=0}^{n-2} \varphi'_{y^{(i)}}(\hat{y})y^{(i+1)}$$

and restrict it to the surface $y^{(n-1)} = \varphi(\hat{y})$:

$$\Psi(\hat{y}) \equiv \mathbf{F}g(\overline{y})\Big|_{y^{(n-1)}=\varphi(\hat{y})} = -f(\hat{y}, \varphi(\hat{y}))$$
$$- \sum_{i=0}^{n-3} \varphi'_{y^{(i)}}(\hat{y})y^{(i+1)} - \varphi'_{y^{(n-2)}}(\hat{y})\varphi(\hat{y}).$$

Since

$$\Psi'_y(\hat{y}) = -f'_y(\hat{y}, \varphi(\hat{y})) - f'_{y^{(n-1)}}(\hat{y}, \varphi(\hat{y}))\varphi'_y(\hat{y})$$
$$- \sum_{j=0}^{n-3} \varphi''_{y^{(j)}y}(\hat{y})y^{(j+1)} - \varphi''_{y^{(n-2)}y}(\hat{y})\varphi(\hat{y}) - \varphi'_{y^{(n-2)}}(\hat{y})\varphi'_y(\hat{y}),$$
$$\Psi'_{y^{(i)}}(\hat{y}) = -f'_{y^{(i)}}(\hat{y}, \varphi(\hat{y})) - f'_{y^{(n-1)}}(\hat{y}, \varphi(\hat{y}))\varphi'_{y^{(i)}}(\hat{y})$$
$$- \sum_{j=0}^{n-3} \varphi''_{y^{(j)}y^{(i)}}(\hat{y})y^{(j+1)} - \varphi'_{y^{(i-1)}}(\hat{y})$$
$$-\varphi''_{y^{(n-2)}y^{(i)}}(\hat{y})\varphi(\hat{y}) - \varphi'_{y^{(n-2)}}(\hat{y})\varphi'_{y^{(i)}}(\hat{y}),\ i = \overline{1, n-2},$$

we have

$$\Psi'_y(0) = -f'_y(0) - f'_{y^{(n-1)}}(0)\varphi'_y(0) - \varphi'_{y^{(n-2)}}(0)\varphi'_y(0)$$
$$= -a_0 - a_{n-1}l_0 - l_{n-2}l_0 = -\xi_0,$$
$$\Psi'_{y^{(i)}}(0) = -f'_{y^{(i)}}(0) - f'_{y^{(n-1)}}(0)\varphi'_{y^{(i)}}(0)$$
$$-\varphi'_{y^{(i-1)}}(0) - \varphi'_{y^{(n-2)}}(0)\varphi'_{y^{(i)}}(0)$$
$$= -a_i - a_{n-1}l_i - l_{i-1} - l_{n-2}l_i = -\xi_i,\ i = \overline{1, n-2}.$$

Hence, since $\Psi(0) = 0$, one gets

$$\Psi(\hat{y}) = -\sum_{i=0}^{n-2} \xi_i y^{(i)} + \alpha(\hat{y})\|\,\hat{y}\,\|$$
$$= \|\,\hat{y}\,\|\Big(\sum_{i=0}^{n-2}(-\xi_i)\frac{y^{(i)}}{\|\,\hat{y}\,\|} + \alpha(\hat{y})\Big),$$

where $\alpha(\hat{y}) \to 0$ as $\|\,\hat{y}\,\| \to 0$.

Let $\xi_* = \min\{-\xi_0, \dots, -\xi_{n-2}\} > 0$, $\nu = (\nu_0, \dots, \nu_{n-2})$, where

$$\nu_i = \frac{y^{(i)}}{\|\,\hat{y}\,\|}, \quad i = 0, \dots, n-2,$$

$$N = \{\nu : \nu > 0,\ \|\,\nu\,\| = 1\}, \quad \overline{N} = \{\nu : \nu \geq 0,\ \|\,\nu\,\| = 1\}.$$

For the function

$$\psi(\nu) = \sum_{i=0}^{n-2} \nu_i$$

we obtain

$$\psi(\nu)|_N \geq \inf_N \psi(\nu) \geq \min_{\overline{N}} \psi(\nu) = \nu_* > 0,$$

because $\overline{N}$ is a compact set and $\psi(\nu)|_{\overline{N}} > 0$.

If $\hat{y} > 0$ one gets

$$\Psi(\hat{y}) \geq \|\,\hat{y}\,\|(\xi_*\psi(\nu) + \alpha(\hat{y})) \geq \|\,\hat{y}\,\|(\xi_*\nu_* + \alpha(\hat{y})).$$

Therefore, there exists $\varepsilon_1 > 0$ for which the inequality $\Psi(\hat{y}) > 0$ holds for all $\hat{y} > 0$ such that $\|\,\hat{y}\,\| < \varepsilon_1$.

Further, since $\varphi(0) = 0$, we have

$$\varphi(\hat{y}) = \sum_{i=0}^{n-2} l_i y^{(i)} + \beta(\hat{y})\|\,\hat{y}\,\|,$$

where $\beta(\hat{y}) \to 0$ as $\|\,\hat{y}\,\| \to 0$. By repeating the previous arguments, without loss of generality, one gets $\varphi(\hat{y}) > 0$ for all $\hat{y} > 0$; $\|\,\hat{y}\,\| < \varepsilon_2$.