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Kalman—Yakubovich—Popov Lemma And Hilbert's 17th Problem

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Abstract: Kalman-Yakubovich-Popov (KYP) lemma is the cornerstone of control theory. It was used in thousands of papers in many areas of automatic control. The new versions and generalizations of KYP lemma emerge in literature every year. The original formulation of KYP lemma claims the equivalence of three statements: 1) fulfillment of so-called frequency-domain inequality, 2) solvability of the KYP linear matrix inequality, and 3) solvability of the Lur'e equation. The equivalence of first two statements was proved by V.A.Yakubovich and is further called Yakubovich statement.
The paper investigates whether the KYP lemma holds when the field of real numbers is replaced by some other ordered field. The necessary and sufficient condition is found for Yakubovich statement to hold in ordered fields. It is shown that Yakubovich statement can hold in such fields when Lur'e equation (and corresponding Riccati equation) has no solution. Based on the statement of Hilbert's 17th problem it is shown that if the matrices in the formulation of Yakubovich statement depend rationally on parameters, then there exists solution of KYP inequality which is also rational function of these parameters. The generalized formulation of Yakubovich statement and Hilbert's 17th problem for abstract ordered fields is presented. It is shown that generalized versions of Yakubovich statement and the statement of Hilbert's 17th problem are equivalent.

Keywords: KYP lemma, frequency domain inequality, LMI, Lur'e equation, SOS, Hilbert's 17th problem

1. INTRODUCTION

The Kalman—Yakubovich—Popov (KYP) lemma is a cornerstone of control theory. It emerged as a tool for absolute stability analysis of nonlinear control systems. Since that it found applications in optimal control, signal processing, adaptive and robust control, control of infinite dimensional systems. The KYP lemma plays an important role as a mathematical bridge between frequency domain and state space methods in control theory.

The first formulation and proof of the lemma was published by Yakubovich (1962) in Russian Mathematical Doklady. Since than thousands of papers have used this result and more then one hundred versions and generalizations was published. The original Yakubovich paper was included into book "Control Theory: Twenty-Five Seminal Papers (1932-1981)" edited by T.Basar. It is interesting to note that besides Yakubovich article this book includes three more papers related to the KYP lemma.

The interest to the KYP lemma has not vanished in recent years. The paper by Iwasaki and Hara (2005)"Generalized KYP lemma: Unified frequency domain inequalities with design applications" got the IEEE Transactions Automatic Control best paper award in 2006. The paper by Tanaka and Langbort (2011)"Symmetric formulation of the Kalman—Yakubovich—Popov lemma and exact losslessness condition" got best student paper award on 50th CDC.

The aim of this article is to explore the algebraic roots of the KYP lemma. While the KYP lemma was always formulated in terms of real and complex numbers, we are considering the possibility to extend the lemma to other fields. A generalized formulation of the lemma for ordered fields is given. The paper is based on our previous result Gusev (2014), where a sufficient condition was found for lemma to hold in abstract ordered field. Now we prove that this condition is in fact a necessary and sufficient one. This implies that a statement of the lemma is equivalent to a statement of generalized version of Hilbert's 17th problem.

2. FORMULATION OF THE PROBLEM.

Let us begin from a classical formulation of the Kalman—Yakubovich—Popov lemma Gelig et al. (1978).

Theorem 1. Let $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times 1}$ and $G\in\mathbb{SM}_{n+1}(\mathbb{R})$. If a pair (A,B) is controllable, then the following statements are equivalent:

1. The inequality

$$\begin{pmatrix} x \\ u \end{pmatrix}^* G \begin{pmatrix} x \\ u \end{pmatrix} \geq 0 \qquad (1)$$

is satisfied for all $\omega\in\mathbb{R}$, $x\in\mathbb{C}^{n\times 1}$, $u\in\mathbb{C}$ such that

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$$\mathbf{i}\omega x = Ax + Bu. \tag{2}$$

2. There exists a matrix $H \in \mathbb{SM}_n(\mathbb{R})$ satisfying the linear matrix inequality

$$G \geq \begin{pmatrix} HA + A^*H & HB \\ B^*H & 0 \end{pmatrix}. \tag{3}$$

3. There exist matrices $H \in \mathbb{SM}_n(\mathbb{R})$, $h \in \mathbb{R}^{n\times 1}$ satisfying the equation

$$G = \begin{pmatrix} HA + A^*H & HB \\ B^*H & 0 \end{pmatrix} + hh^*. \tag{4}$$

Here, $\mathbb{R}$ ($\mathbb{C}$) is the field of real (complex) numbers; $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) is the space of real (complex) $m \times n$ matrices; $\mathbb{SM}_n(\mathbb{R})$ is the space of symmetrical real matrices of size n; $\mathbf{i}$ is the imaginary unit, and the operation * is the transposition in the real case and conjunction in the complex one.

Statement **1** is called the frequency condition, Eq. (4) is the Lur'e equation. Equivalence of **1** and **2** was proved by Yakubovich (1962), equivalence of **1** and **3**, by Kalman (1963). For a detailed history of origination and proof of the lemma the readers are referred to Gusev and Likhtarnikov (2006).

The Kalman—Yakubovich—Popov lemma is closely connected with the question of solution existence for the algebraic Riccati equation and inequality. Let us represent the matrix G in the block form as $G = \begin{pmatrix} G_{xx} & G_{xu} \\ G_{ux} & G_{uu} \end{pmatrix}$, where $G_{xx} \in \mathbb{SM}_n(\mathbb{R})$, $G_{xu} \in \mathbb{R}^{n\times 1}$, $G_{ux} \in \mathbb{R}^{1\times n}$, and $G_{uu} \in \mathbb{R}$. If $G_{uu} > 0$, then the statements **2** and **3** are equivalent respectively to the following statements:

2′. There exists a matrix $H \in \mathbb{SM}_n(\mathbb{R})$ satisfying the Riccati inequality

$$G_{xx} - HA - A^*H - (G_{xu} - HB)G_{uu}^{-1}(G_{xu} - HB)^* \geq 0. \tag{5}$$

3′. There exists a matrix $H \in \mathbb{SM}_n(\mathbb{R})$ satisfying the algebraic Riccati equation

$$G_{xx} - HA - A^*H - (G_{xu} - HB)G_{uu}^{-1}(G_{xu} - HB)^* = 0. \tag{6}$$

The aim of the paper is to explore the mathematical foundations of KYP lemma. To this end let us replace the field of real numbers $\mathbb{R}$ in the formulation of lemma by an abstract ordered field $\mathbf{F}$.

Kalman (1970) had shown that the KYP lemma holds for any real closed field $\mathbf{F}$. This is very tough condition on the field. The question arises: can the KYP Lemma be fulfilled under less restrictive assumption on the field $\mathbf{F}$? It can be shown that the answer is negative for a standard formulation of KYP lemma which includes equivalence of three statements. But we shall show that the equivalence of first two statements of KYP lemma holds for many other fields. Because the equivalence of two first statements of KYP lemma was proved by V.A.Yakubovich we shall further refer to this part of the KYP lemma as Yakubovich lemma.

To illustrate the above claims let us formulate Yakubovich lemma for the field of rational numbers $\mathbb{Q}$.

Theorem 2. Let $A \in \mathbb{Q}^{n\times n}$, $B \in \mathbb{Q}^{n\times 1}$ and $G \in \mathbb{SM}_{n+1}(\mathbb{Q})$. If the pair (A, B) is controllable, then the following statements are equivalent:

1. The inequality (1) is fulfilled for all $\omega \in \mathbb{Q}$, $x \in \mathbb{Q}_c^{n\times 1}$, $u \in \mathbb{Q}_c$ satisfying (2).

2. There exists a matrix $H \in \mathbb{SM}_n(\mathbb{Q})$ satisfying (3).
Here $\mathbb{Q}_c$ is a complex extension of the field $\mathbb{Q}$.

It should be noted that in this case the third statement of the KYP lemma is not equivalent to first two statements. Consider an example. Let $G = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, $A = 1$, $B = 1$. Then the statements of the KYP lemma (formulated in terms of Riccati inequality and equation) take the following form:

1. $2 - |\mathbf{i}\omega - 1|^{-2} \geq 0 \ \forall \omega \in \mathbb{Q}$.
2′. $H^2 + 4H + 2 \leq 0$
3′. $H^2 + 4H + 2 = 0$

In case when the field of real numbers is considered all three statements hold. But if the solution H has to belong to the field of rational numbers, first two statements hold, but third does not hold, since the solutions of the Riccati equation are irrational $H_{1,2} = -2 \pm \sqrt{2}$.

The proof of the theorem will be given in the next section.

3. YAKUBOVICH LEMMA IN ORDERED FIELDS

Let $\mathbf{F}$ be ordered field. Consider in the field $\mathbf{F}$ a new partial order $\succeq$, defining the totally non-negative elements as sums of squares, i.e. $a \succeq 0 \iff a = \sum_j c_j^2$. This order induces the partial order in the space of symmetric matrices $\mathbb{SM}_n(\mathbf{F})$: $A \succeq 0$, if $\forall x \in \mathbf{F}^{n\times 1}$ $x^*Ax \succeq 0$.

Yakubovich statement.
For all n, $A \in \mathbf{F}^{n\times n}$, $B \in F^{n\times 1}$ and $G \in \mathbb{SM}_{n+1}(\mathbf{F})$, if the pair (A, B) is controllable, then the following statements are equivalent:

1. For all $x \in \mathbf{F}_c^n, u \in \mathbf{F}_c, \omega \in \mathbf{F}$ such that $\mathbf{i}\omega x = Ax + Bu$ the inequality

$$\begin{pmatrix} x \\ u \end{pmatrix}^* G \begin{pmatrix} x \\ u \end{pmatrix} \succeq 0$$

holds.

2. There exists $H \in \mathbb{SM}_n(\mathbf{F})$ satisfying the LMI

$$G \succeq \begin{pmatrix} HA + A^*H & HB \\ B^*H & 0 \end{pmatrix}.$$

Here $\mathbf{F}_c$ is a complex extension of the field $\mathbf{F}$.
Our purpose is to define for what fields the Yakubovich statement holds.

Definition 1. The ordered field $\mathbf{F}$ has sums of squared polynomials property (SOSP for short), if every polynomial in one variable $p \in \mathbf{F}[\lambda]$, that satisfies the inequality $p(\lambda) \succeq 0$ for all $\lambda \in \mathbf{F}$, can be represented as sum of squares of polynomials.

Theorem 3. Yakubovich statement is fulfilled in the field if and only if this field has SOSP property.

The proof of theorem is given in appendix.