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# An Extended Kalman-Yakubovich-Popov Lemma for Positive Systems

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**Abstract:** An extended Kalman-Yakubovich-Popov Lemma for positive systems is proved, which generalizes earlier versions in several respects: Non-strict inequalities are treated. Matrix assumptions are less restrictive. Moreover, a new equivalence is introduced in terms of linear programming rather than semi-definite programming. As a complement, we also prove that a symmetric Metzler matrix with $m$ non-zero entries above the diagonal is negative semi-definite if and only if it can be written as a sum of $m$ negative semi-definite matrices, each of which has only four non-zero entries.



## 1. INTRODUCTION

The Kalman-Yakubovich-Popov (KYP) Lemma is recognized as one of the cornerstones of modern systems theory Gusev and Likhtarnikov (2006) and more than fifty years after its conception it is still a topic of active research Iwasaki and Hara (2005); Gusev (2009). The result creates a bridge between frequency domain properties and state space descriptions of linear systems. For example, it is commonly used to convert frequency domain control design problems into convex optimization.

This paper is devoted to a strengthening of the KYP lemma for the special case of positive systems. The theory of positive systems and nonnegative matrices has a long history, dating back to the Perron-Frobenius Theorem in 1912. A classic book is Berman and Plemmons (1994). The topic has recently gained increasing attention in the control literature Farina and Rinaldi (2000); Kaczorek (2002), especially in the context of distributed control Rantzer (2011). It was recently discovered in Tanaka and Langbort (2011) that for positive systems the KYP lemma can be stated in terms of a diagonal matrix variable rather than a general symmetric one. This simplifies applications considerably. For example, it was used in Tanaka and Langbort (2011) for $H_\infty$ optimal design of distributed controllers. A corresponding statement of the KYP lemma for discrete time positive systems was given in Najson (2013). Related criteria for $L_1$-gain and $L_\infty$-gain were studied in Ebihara et al. (2011); Briat (2013).

Building on the earlier conference paper Rantzer (2012), this paper generalizes past results in several respects. The new statements are given in section 3 for both continuous and discrete time positive systems, followed by a complementing theorem that simplifies verification of semi-definiteness for large Metzler matrices. Proofs are given in section 4.

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## 2. NOTATION

The inequality $X > 0$ ($X \geq 0$) means that all elements of the matrix (or vector) $X$ are positive (nonnegative). For a symmetric matrix $X$, the inequality $X \succeq 0$ means that the matrix is positive semidefinite. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Hurwitz* if all eigenvalues have negative real part. It is *Schur* if all eigenvalues are strictly inside the unit circle. Finally, the matrix is said to be *Metzler* if all off-diagonal elements are nonnegative.

## 3. THE KYP LEMMA FOR POSITIVE SYSTEMS

*Theorem 1.* Let $A \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz, while $B \in \mathbb{R}_+^{n\times m}$ and the pair $(A, B)$ is controllable. Suppose that $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$ is nonnegative in all entries, except for the last $m$ diagonal elements. Then the following statements are equivalent:

**(1.1)** For all $\omega \in [0, \infty]$ it is true that

$$\begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix} \preceq 0$$

**(1.2)** $$\begin{bmatrix} -A^{-1}B \\ I \end{bmatrix}^T M \begin{bmatrix} -A^{-1}B \\ I \end{bmatrix} \preceq 0.$$

**(1.3)** There exists a diagonal $P \succeq 0$ such that

$$M + \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} \preceq 0$$

If all inequalities are replaced by strict ones, then the equivalences hold even without the controllability assumption and the following statement is also equivalent:

**(1.4)** There exist $x, p, u > 0$ with $Ax + Bu < 0$ and

$$M \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} p < 0 \qquad (1)$$


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*Remark 1.* For $A = -1$, $B = 0$, $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, condition (1.1) holds, but not (1.3). This shows that controllability is essential when the inequalities are non-strict.

*Remark 2.* Our formulation of the KYP lemma for positive systems extends earlier versions of Tanaka and Langbort (2011) and Najson (2013) in three respects: Non-strict inequalities, more general $M$ and statement (1.4).

As expected, there is also an analogous discrete time result:

*Theorem 2.* Let $A \in \mathbb{R}_+^{n\times n}$ be Schur, while $B \in \mathbb{R}_+^{n\times m}$ and the pair $(A, B)$ is controllable. Suppose that all entries of the symmetric matrix $M \in \mathbb{R}^{(n+m)\times(n+m)}$ are nonnegative, except for the last $m$ diagonal elements. Then the following statements are equivalent:

**(2.1)** For all $\omega \in [0, \infty]$ it is true that
$$\begin{bmatrix} (e^{i\omega}I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (e^{i\omega}I - A)^{-1}B \\ I \end{bmatrix} \preceq 0$$

**(2.2)** $\begin{bmatrix} (I - A)^{-1}B \\ I \end{bmatrix}^T M \begin{bmatrix} (I - A)^{-1}B \\ I \end{bmatrix} \preceq 0.$

**(2.3)** There exists a diagonal $P \succeq 0$ such that
$$M + \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} \preceq 0$$

If all inequalities are replaced by strict ones, then the equivalences hold even without the controllability assumption and the following statement is also equivalent:

**(2.4)** There exist $x, p \in \mathbb{R}_+^n$, $u \in \mathbb{R}_+^m$ with $x > Ax + Bu$ and
$$M \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} A^T - I \\ B^T \end{bmatrix} p < 0$$

Theorem 1 and Theorem 2 are fundamentally different from the standard KYP lemma Rantzer (1996) in that the matrix variable $P$ is diagonal. Hence the number of variables grows linearly rather than quadratically with $n$. However, verfication of negative semi-definiteness could still be a source of higher complexity. The next theorem shows that this condition can also be simplified.

*Theorem 3.* A symmetric Metzler matrix with $m$ non-zero entries above the diagonal is negative semi-definite if and only if it can be written as a sum of $m$ negative semi-definite matrices, each of which has only four non-zero entries.

The following illustrates how the Theorem 3 can be used to verify the validity of a Lyapunov function candidate.

**Example 1.** All eigenvalues of the matrix
$$A = \begin{bmatrix} 0 & 0.7 & 0 & 0.6 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0.6 & 0 \\ 0.9 & 0 & 0.7 & 0 \end{bmatrix}$$
are inside the unit circle. By (Farina and Rinaldi, 2000, Theorem 15), this is equivalent to existence of a diagonal solution $P \succ 0$ to the Lyapunov inequality $A^TPA \prec P$. To see that $P = \text{diag}\{4, 10, 10, 2\}$ is a feasible solution we may use Theorem 3 and verify that
$$A^TPA - P = \begin{bmatrix} -0.78 & 0 & 1.26 & 0 \\ 0 & 0 & 0 & 0 \\ 1.26 & 0 & -2.8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1.66 & 1.8 & 0 \\ 0 & 1.8 & -2.62 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5.48 & 0 & 1.68 \\ 0 & 0 & 0 & 0 \\ 0 & 1.68 & 0 & -0.56 \end{bmatrix}.$$
where each of the four nonzero submatrices is clearly negative definite. □

Notice that the number of scalar variables, inequalities and nontrivial equalities involved in the verification grows linearly with the number of nonzero entries in $A^TPA - P$. Of course, Theorem 3 can be used in a similar way to verify the inequalities in (1.2), (1.3), (2.2) and (2.3).

## 4. PROOFS

The proofs of Theorem 1-3 will be based on the following result inspired by (Kim and Kojima, 2003, Theorem 3.1). The notation $\text{diag}(NZ) \geq 0$ means that the diagonal elements of $NZ$ are nonnegative.

*Theorem 4.* Let $M$ and $N$ be square and Metzler. $M$ is also symmetric. Then the following equality holds:
$$\begin{array}{lll} \max & z^TMz & = \max \quad \text{trace}(MZ) \\ z \in \mathbb{R}_+^n & Nz \geq 0 & Z \succeq 0 \;\; \text{diag}(NZ) \geq 0 \\ & z^Tz \leq 1 & \qquad\quad \text{trace}\, Z \leq 1 \end{array} \tag{2}$$

The right hand side value in (2) remains the same if the condition $Z \succeq 0$ is relaxed to $Z \in \mathbb{P}$, where $\mathbb{P}$ is the set of symmetric matrices $(z_{ij}) \in \mathbb{R}^{n\times n}$ satisfying $z_{ii} \geq 0$ and $z_{ij}^2 \leq z_{ii}z_{jj}$ for all $i, j$.

*Remark 3.* The problem on the right of (2) is convex and readily solvable by semidefinite programming.

*Remark 4.* Theorem 4 generalizes straightforwardly to problems with several contraints of the form $z^TM_kz \geq m_k$, $N_kz \geq 0$, where $M_k$ and $N_k$ are Metzler.

**Proof of Theorem 4.** For every $z$ satisfying the constraints on the left hand side of (2), the expression $Z = zz^T$ gives a matrix satisfying the constraints on the right hand side. This shows that the right hand side of (2) is at least as big as the left. On the other hand, let $Z = (z_{ij})$ be a positive semidefinite matrix. In particular, the diagonal elements $z_{11}, \ldots, z_{nn}$ are non-negative and $z_{ij} \leq \sqrt{z_{ii}z_{jj}}$. Define $z = (\sqrt{z_{11}}, \ldots, \sqrt{z_{nn}})$. Then the matrix $zz^T$ has the same diagonal elements as $Z$, but has off-diagonal elements $\sqrt{z_{ii}z_{jj}}$ instead of $z_{ij}$. The fact that $zz^T$ has off-diagonal elements at least as big as those of $Z$, together with the assumption that the matrix $M$ is Metzler, gives $z^TMz \geq \text{trace}(MZ)$. Let $N_k$ denote the $k$th row of the matrix $N$. The inequality $\text{diag}(NZ) \geq 0$ together with the fact that $N$ is Metzler implies that $\text{diag}(Nzz^T) \geq 0$ and therefore $z_kN_kz \geq 0$ for all $k$. The inequality $z_kN_kz \geq 0$ gives $N_kz \geq 0$ since $z_k \geq 0$ and $N$ is Metzler. This shows that the left hand side of (2) is at least as big as the right, so the equality (2) follows. The argument remains valid if $Z$ is not positive definite but $Z \in \mathbb{P}$, so the proof is complete.