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IQC arguments for analysis of the 3-state Moore-Greitzer compressor system

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Abstract: The Integral Quadratic Constraint (IQC) framework developed by Professor Yakubovich and his co-workers, see Yakubovich et.al. (2004), is one of few available constructive tools for establishing robust stability of nonlinear systems. An explicit format of stability conditions, procedures for computing a Lyapunov function and developed libraries of IQCs for common nonlinearities in dynamics, all together have made the approach unique and at the same time so to say automatic for recovering stability conditions for many applications: in the process of analyzing a dynamical system, an engineer is just required to search for a sufficiently rich set of IQCs describing nonlinearities in the dynamics so that such nonlinearities can be substituted in analysis by quadratic constraints, which they satisfy. The power of the methodology becomes also its weak part in an analysis of concrete systems. Searching IQCs is the difficult task in new examples, where a lack of a rich set of IQCs for concrete nonlinearities makes the method inconclusive or too rough to detect (in)-stability. The paper is aimed at a discussion of such an example of a nonlinear dynamical system (the classical 3-state Moore-Greitzer compressor model) augmented with the dynamical feedback controller, whose parameters should be adjusted to meet a stability condition. The closed-loop system has several nonlinearities and searching the corresponding IQCs to meet the stability conditions for this example is rather involved. To overcome the problem, we have previously described by different methods a set of parameters for which any solution of the closed loop system, if bounded, will converge to the origin and that the origin is locally asymptotically stable. However, the proof is incomplete without showing a boundedness of all solutions. To solve the task we have re-used some of the IQC framework ideas, where the method has been utilized and the corresponding IQCs have been found only for unbounded trajectories, if they would be present in closed loop system. The arguments have allowed completing the proof of stability and illustrating deliberate use of the IQC framework aimed at analysis of behavior of specific trajectories.

Keywords: Nonlinear control systems, Integral Quadratic Constraint, 'Frequency' condition, 3-state Moore-Greizer compressor model, dynamic feedback controller design

1. INTRODUCTION AND PRELIMINARIES

The 3-state Moore-Greitzer (3MG) model of the compressor dynamics, see Moore (1984); Greitzer and Moore (1986); Moore and Greitzer (1986); Gravdahl and Egeland (1999) and others, is the nonlinear control system that has been approached and explored by control scientists for two decades as a challenging bench-mark example for feedback design and analysis of nonlinear closed loop systems. The system dynamics have only three state variables, whose time evolution is determined by the equations (Moore and Greitzer , 1986, Eqs. (59), (60), and (61))

$$\frac{d}{dt}\phi = \frac{3}{2}\phi - \psi + \frac{1}{2}\left[1 - (1+\phi)^3\right] - 3R(1+\phi) \quad (1)$$

$$\frac{d}{dt}\psi = \frac{1}{\beta^2}\left(\phi - u\right) \quad (2)$$

$$\frac{d}{dt}R = -\sigma R^2 - \sigma R\left(2\phi + \phi^2\right), \quad R(0) \geq 0 \quad (3)$$

Here $\phi(\cdot)$ and $\psi(\cdot)$ denote deviations of the averaged flow and the total-to-static pressure-rise coefficients from their nominal values, respectively; u is defined by deviation of the coefficient of the inverse throttle characteristic function from a nominal value; and t is a scaled time measured in radians of travel of the compressor wheel; σ and β are positive constant parameters. The dynamics of the variable $R(\cdot)$, known as stall, is often treated as a perturbation to (1)–(2).

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If $R(0) = 0$, then it is readily seen that the stall variable $R(t)$ will be kept zero for all $t \geq 0$ irrespective of the control input, and the 3MG model (1)-(3) is reduced to its subsystem

$$\begin{aligned} \tfrac{d}{dt}\phi &= \tfrac{3}{2}\phi - \psi + \tfrac{1}{2}\left[1 - (1+\phi)^3\right] \\ \tfrac{d}{dt}\psi &= \tfrac{1}{\beta^2}(\phi - u) \end{aligned} \tag{4}$$

known as *surge dynamics*. The nonlinearity of (1) and (4)

$$W^{\{\phi\}}(v) := 1 - (1+v)^3 \tag{5}$$

is called a *compressor characteristic*. It is clearly dependent on the specific design of the compressor and determined as an outcome of experimental studies following an appropriate fitting of the data collected (often chosen as a 3rd order polynomial).

The control task is to design a feedback controller for the system[1] (1)-(3) such that it ensures global asymptotic stability of the origin.

In this short remark we will focus on a sub-task and will search for sufficient conditions ensuring a boundedness of closed-loop system solutions when the 3MG model is augmented with a dynamical feedback controller from the following parametric class

$$\begin{aligned} u &= \phi - \beta^2\left\{\lambda_\phi\phi + \lambda_\psi\psi + \lambda_z z + \alpha\left[1 - (1+\phi)^3\right]\right\} \\ \tfrac{d}{dt}z &= -\phi. \end{aligned} \tag{6}$$

Here $\{\lambda_\phi,\, \lambda_\psi,\, \lambda_z,\, \alpha\}$ are design parameters to be determined. It is convenient to re-write the closed loop system dynamics (1)-(3), (6) as an interconnection of the stall dynamics (3) and the controlled surge

$$\frac{d}{dt}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} = \mathcal{A}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} + \mathcal{B}W^{\{\phi\}}(\phi) + \mathcal{B}_R W^{\{R\}}(R,\phi) \tag{7}$$

Here the nonlinearity $W^{\{\phi\}}(\cdot)$ is defined in (5); coefficients of the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}_R$ are

$$\mathcal{A} = \begin{bmatrix}1.5 & -1 & 0\\ \lambda_\phi & \lambda_\psi & \lambda_z\\ -1 & 0 & 0\end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix}0.5\\ \alpha\\ 0\end{bmatrix}, \quad \mathcal{B}_R = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \tag{8}$$

and $W^{\{R\}}(\cdot,\cdot)$ is an abbreviation for the scalar nonlinearity

$$W^{\{R\}}(R,v) := -3\cdot R\cdot(1+v) \tag{9}$$

with $W^{\{R\}}(0,v) \equiv 0,\ \forall v$.

Since the stall variable $R(t)$ is zero for all positive time moments irrespective of the control input u provided that $R(0) = 0$, then stabilization of the 3MG-model (1)-(3) implies stabilization of the surge subsystem (4). Therefore, by necessity the parameters $\{\lambda_\phi,\, \lambda_\psi,\, \lambda_z,\, \alpha\}$ of the controller (6) should be chosen in such a way that the dynamical system

$$\frac{d}{dt}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} = \mathcal{A}\begin{bmatrix}\phi\\ \psi\\ z\end{bmatrix} + \mathcal{B}W^{\{\phi\}}(\phi) \tag{10}$$

is asymptotically stable.

[1] and therefore to design a feedback controller for the surge subsystem (4).

The stability conditions of (10) and corresponding ranges of controller parameters can be efficiently computed based on the circle criterion. Indeed, the nonlinearity (5) satisfies the QC

$$\begin{aligned} \mathcal{G}_1\left[v, W^{\{\phi\}}(v)\right] &= W^{\{\phi\}}(v)\cdot(-v) - \tfrac{3}{4}v^2 \\ &= v^2\left(v + \tfrac{3}{2}\right)^2 \geq 0 \end{aligned} \tag{11}$$

Furthermore, solutions of the closed-loop system (10) satisfies an IQC as formulated below

Lemma 1. Consider the closed-loop system (10) and suppose $[0, \tau^{max})$ is the maximum interval of existence of a solution

$$X(t) = [\phi(t); \psi(t); z(t)]$$

of (10), where $\tau^{max} > 0$ can be finite or not. Then, only the following two cases are possible:

(1) There exists a sequence $\{t_k\}_{k=1}^{\infty}$ of time instants with $\lim_{k\to\infty} t_k = \tau^{max}$ such that the integrals of the quadratic form $\mathcal{G}_1[\cdot]$ defined in Eq. (11) computed along ϕ-component of $X(t)$ are all positive, i.e.

$$\int_{t_k}^{t_{k+1}} \mathcal{G}_1\left[\phi(t), W^{\{\phi\}}(\phi(t))\right] dt > 0, \quad k = 1,\, 2,\, \ldots \tag{12}$$

(2) Along this solution,

$$\text{either} \quad \phi(t) \equiv 0 \quad \text{or} \quad \phi(t) \equiv -3/2 \tag{13}$$

Moreover, the integral in (12) of the quadratic form $\mathcal{G}_1\big[\phi(t), W^{\{\phi\}}(\phi(t))\big]$ identically equals zero for any $t \in [0, \tau^{max})$. ■

Proof is trivial. The relations (12), (13) established for any motion of the closed-loop system (10) permit to use the IQC framework arguments and formulate the next stability test.

Lemma 2. Consider the closed-loop system (10). Suppose that:

(1) There is a matrix $P = P^T$ such that the following inequality holds for all x and w_1

$$2x^T P(\mathcal{A}x + \mathcal{B}w_1) + \mathcal{G}_1[\mathcal{C}x, w_1] \leq 0 \tag{14}$$

with $\mathcal{C} = [1, 0, 0]$;

(2) The matrix $\left(\mathcal{A} - \tfrac{3}{4}\mathcal{B}\mathcal{C}\right)$ is Hurwitz and the pair $[\mathcal{C}, \mathcal{A}]$ is observable.

Then, the origin of the nonlinear system (10) is locally exponentially stable and globally asymptotically stable. ■

Proof of Lemma 2 is given in Appendix A.

As well known, the validity of (14) can be equivalently reformulated as the validity of the corresponding 'frequency' condition. For the closed-loop system (10) it will provide explicit relations between the controller parameters $\{\lambda_\phi,\, \lambda_\psi,\, \lambda_z,\, \alpha\}$ associated with the guaranteed asymptotic stability.

Unfortunately, even robust stabilization of the surge dynamics does not in general imply the stabilization of the 3MG model with nontrivial stall dynamics. However, this is the case for the parametric set of controllers (6) analyzed in the paper. The complete proof of the result goes beyond the scope of the paper, and, as the main contribution, we