Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-11 (2015) 264–267

Infinite-Time Optimal Control of Nonlinear Continuous-Time Systems with Nonlinear Feedback Gains

Jakub Bernat, Slawomir Stepien and Paulina Superczynska

Chair of Computer Engineering,Poznan University of Technology, Poznan, Poland (e-mail: slawomir.stepien@put.poznan.pl)

Abstract: The optimal control problem for continuous-time nonlinear systems with nonlinearity in state is formulated and solved. A solution to the problem using state-dependent Riccati equation method (SDRE) is established, procedure for solving the problem is proposed and illustrated with a numerical example.

Keywords: Nonlinear systems, Optimal feedback control, State-dependent Riccati equations.

1. INTRODUCTION

Optimal control of electric motors is an area of active research worldwide. Efforts concentrate on finding solutions which guarantee high immunity of the drive system in terms of realizing prescribed trajectories while minimizing associated errors in position, velocity or torque [1,2,5,6]. The literature coverage in this field is immense, but some specific topics deserve to be mentioned, in particular the optimal control theory related to minimization of energy delivered to the drive [6,7,9]. In this area, linear quadratic regulation is a well-established and generally accepted for the synthesis of control laws for linear systems [7,8]. However, a variety of nonlinear models can be found in engineering, economics, social sciences, biology and medicine, etc [9]. In literature exist solutions for optimal control of nonlinear systems, but the best suited approaches for designing nonlinear controllers are state-dependent Riccati equation approach and frozen Riccati equation method [4,6,10,11].

In this paper, as a new contribution in area related to the optimal control of nonlinear systems, the control problem for continuous-time nonlinear systems with nonlinearity in state is formulated and solved. The approach to the optimal control using nonlinear feedback controller enables accurate determination of optimal control allowing optimal control theory to be employed [4,10,11]. This is the focus of this paper.

2. OPTIMAL CONTROL PROBLEM

Consider the continuous-time nonlinear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}\boldsymbol{\Psi}(\mathbf{x}) + \mathbf{B}\mathbf{U} \tag{1}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{U} \in \Re^m$ are the state and input vectors, respectively, $\boldsymbol{\Psi}(\mathbf{x}) \in \Re^{p\times 1}$ is a nonlinear function of $\mathbf{x}$ and $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{F} \in \Re^{n\times p}$, $\mathbf{B} \in \Re^{n\times m}$. We wish to find an admissible control $\mathbf{U} \in \Re^m$ that minimizes performance index

$$J(\mathbf{U}) = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{U}^T\mathbf{R}\mathbf{U}\right)dt \tag{2}$$

where $\mathbf{Q} \in \Re^{n\times n}$ is a symmetric semi positive-definite matrix, $\mathbf{R} \in \Re^{m\times m}$ is a symmetric positive-definite matrix.

The optimal control problem for the nonlinear continuous-time systems (1) can be stated as follows. Given matrices $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{B} \in \Re^{n\times m}$, $\mathbf{F} \in \Re^{n\times p}$ and $\mathbf{Q} \in \Re^{n\times n}$, $\mathbf{R} \in \Re^{m\times m}$ of the performance index (2), find a control $\mathbf{U} \in \Re^m$ for $t \in [t_0, \infty]$ that controls the system state vector from $\mathbf{x}_0$ to $\mathbf{x}_\infty$ while minimizing the performance index (2).

Theorem 1. *Let* $\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{U}^T\mathbf{R}\mathbf{U}$ *and* $\mathbf{A}\mathbf{x} + \mathbf{F}\boldsymbol{\Psi}(\mathbf{x}) + \mathbf{B}\mathbf{U}$ *be continuously differentiable functions of each of their arguments. Suppose that* $\mathbf{U}^* \in C[t_0,\infty]$ *is an optimal control for the functional* $J(\mathbf{U}): C[t_0,\infty] \to \Re$ *defined as follows: If* $\mathbf{U} \in C[t_0,\infty]$*, then considering functional (2),* $\mathbf{x}$ *denotes the unique solution to the nonlinear differential equation (1).*

To solve the problem, we define the Hamiltonian

$$H = \frac{1}{2}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{U}^T\mathbf{R}\mathbf{U}\right) + \mathbf{p}^T\left(\mathbf{A}\mathbf{x} + \mathbf{F}\boldsymbol{\Psi}(\mathbf{x}) + \mathbf{B}\mathbf{U}\right) \tag{3}$$

and **Theorem 1** can be equivalently expressed in the following form.

Theorem 2. *Let* $\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{U}^T\mathbf{R}\mathbf{U}$ *and* $\mathbf{A}\mathbf{x} + \mathbf{F}\boldsymbol{\Psi}(\mathbf{x}) + \mathbf{B}\mathbf{U}$ *be continuously differentiable functions of each of their arguments. If* $\mathbf{U}^* \in C[t_0,\infty]$ *is an optimal control for the functional (2) subject to the state equation (1) and if* $\mathbf{x}^*$ *denotes the corresponding state, then there exists a* $\mathbf{p}^* \in C^1[t_0,\infty]$ *such that*

$$\frac{\partial H}{\partial \mathbf{U}}\left(\mathbf{p}^*, \mathbf{x}^*, \mathbf{u}^*, t\right) = \mathbf{0} \text{ for } t \in [t_0,\infty] \tag{4}$$

and

$$\dot{\mathbf{p}}^* = -\frac{\partial H}{\partial \mathbf{x}}\left(\mathbf{p}^*, \mathbf{x}^*, \mathbf{u}^*, t\right) \text{ for } t \in [t_0,\infty] \text{ and } \mathbf{p}^*(\infty) = 0\,. \tag{5}$$

where $\mathbf{p}^*$ *is the co-state function and (5) is the adjoint differential equation.*

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.09.195

From **Theorem 2**, it follows that any optimal input $\mathbf{U}^*$ and the corresponding state $\mathbf{x}^*$ satisfies (4) that is

$$\frac{\partial H}{\partial \mathbf{U}} = \mathbf{R}\mathbf{U}^* + \mathbf{B}^T\mathbf{p}^* = \mathbf{0}. \tag{6}$$

Thus optimal control is

$$\mathbf{U}^* = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{p}^* \tag{7}$$

and adjoint differential equation

$$\dot{\mathbf{p}}^* = -\frac{\partial H}{\partial \mathbf{x}} = -\left[\mathbf{A}^T + \left[\frac{\partial \mathbf{\Psi}}{\partial \mathbf{x}}\right]^T \mathbf{F}^T\right]\mathbf{p}^* - \mathbf{Q}\mathbf{x}. \tag{8}$$

Consequently, we have nonlinear time-invariant differential system of equation:

$$\begin{aligned} \dot{\mathbf{x}}^* &= \mathbf{A}\mathbf{x}^* + \mathbf{F}\mathbf{\Psi}(\mathbf{x}^*) - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{p}^* \\ \dot{\mathbf{p}}^* &= -\frac{\partial H}{\partial \mathbf{x}} = -\left[\mathbf{A}^T + \left[\frac{\partial \mathbf{\Psi}}{\partial \mathbf{x}}\right]^T \mathbf{F}^T\right]\mathbf{p}^* - \mathbf{Q}\mathbf{x} \end{aligned} \tag{9}$$

for $t \in [t_0, \infty]$, $\mathbf{x}^*(t_0) = \mathbf{x}_0$, $\mathbf{\Psi}(\mathbf{x}_0) = \mathbf{\Psi}_0$ and $\mathbf{p}^*(\infty) = 0$.

This system of equation deals with initial $\mathbf{x}^*(t_0) = \mathbf{x}_0$, $\mathbf{\Psi}(\mathbf{x}_0) = \mathbf{\Psi}_0$ and final condition $\mathbf{p}^*(\infty) = 0$.

Theorem 3. *Let $\mathbf{p}^*$ be a combination of linear and nonlinear part of the system (1)*

$$\mathbf{p}^* = \mathbf{K}_1(\mathbf{x}^*)\mathbf{x}^* + \mathbf{K}_2(\mathbf{x}^*)\mathbf{\Psi}(\mathbf{x}^*), \tag{10}$$

and let $\mathbf{x}^*$ be the solution of nonlinear state equation

$$\dot{\mathbf{x}}^* = \left[\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_1(\mathbf{x}^*)\right]\mathbf{x}^* + \left[\mathbf{F} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_2(\mathbf{x}^*)\right]\mathbf{\Psi}(\mathbf{x}^*) \tag{11}$$

for $t \in [t_0, \infty]$, $\mathbf{x}^*(t_0) = \mathbf{x}_0$, $\mathbf{\Psi}(\mathbf{x}_0) = \mathbf{\Psi}_0$ then vectors $\mathbf{x}^*$, $\mathbf{p}^*$ are the unique solution of (9) where matrix gains $\mathbf{K}_1(\mathbf{x}) \in \Re^{n\times n}$, $\mathbf{K}_2(\mathbf{x}) \in \Re^{n\times p}$ are solutions of following nonlinear equation system (state dependent Riccati equations):

$$\begin{aligned} &\mathbf{K}_1(\mathbf{x})\mathbf{A} + \left[\mathbf{A}^T + \mathbf{J}^T(\mathbf{x})\mathbf{F}^T\right]\mathbf{K}_1(\mathbf{x}) - \mathbf{K}_1(\mathbf{x})\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_1(\mathbf{x}) + \\ &\mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\mathbf{A} - \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_1(\mathbf{x}) + \mathbf{Q} = \mathbf{0} \end{aligned} \tag{12}$$

$$\begin{aligned} &\mathbf{K}_1(\mathbf{x})\mathbf{F} - \mathbf{K}_1(\mathbf{x})\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_2(\mathbf{x}) - \\ &\mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_2(\mathbf{x}) + \\ &\mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\mathbf{F} + \left[\mathbf{A}^T + \mathbf{J}^T(\mathbf{x})\mathbf{F}^T\right]\mathbf{K}_2(\mathbf{x}) = \mathbf{0} \end{aligned}, \tag{13}$$

where $\mathbf{J}(\mathbf{x}) = \dfrac{\partial \mathbf{\Psi}}{\partial \mathbf{x}} \in \Re^{p\times n}$.

In above theorem, the nonlinear system of equation is obtained equating adjoint differential equation (8)

$$\begin{aligned} \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} &= -\left[\mathbf{A}^T + \left[\frac{\partial \mathbf{\Psi}}{\partial \mathbf{x}}\right]^T \mathbf{F}^T\right]\mathbf{p} - \mathbf{Q}\mathbf{x} = \\ &-\left[\left[\mathbf{A}^T + \mathbf{J}^T(\mathbf{x})\mathbf{F}^T\right]\mathbf{K}_1(\mathbf{x}) + \mathbf{Q}\right]\mathbf{x} - \\ &\left[\mathbf{A}^T + \mathbf{J}^T(\mathbf{x})\mathbf{F}^T\right]\mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x}) \end{aligned} \tag{14}$$

and differential form of (10)

$$\begin{aligned} \dot{\mathbf{p}} &= \dot{\mathbf{K}}_1(\mathbf{x})\mathbf{x} + \dot{\mathbf{K}}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x}) + \mathbf{K}_1(\mathbf{x})\dot{\mathbf{x}} + \mathbf{K}_2(\mathbf{x})\dot{\mathbf{\Psi}}(\mathbf{x}) = \\ &= \dot{\mathbf{K}}_1(\mathbf{x})\mathbf{x} + \dot{\mathbf{K}}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x}) + \mathbf{K}_1(\mathbf{x})\dot{\mathbf{x}} + \mathbf{K}_2(\mathbf{x})\frac{\partial \mathbf{\Psi}}{\partial \mathbf{x}}\dot{\mathbf{x}} = \\ &\dot{\mathbf{K}}_1(\mathbf{x})\mathbf{x} + \dot{\mathbf{K}}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x}) + \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\dot{\mathbf{x}} = \\ &\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\left[\mathbf{A}\mathbf{x} + \mathbf{F}\mathbf{\Psi}(\mathbf{x}) + \mathbf{B}\mathbf{U}\right] = \\ &\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\times \\ &\left[\mathbf{A}\mathbf{x} + \mathbf{F}\mathbf{\Psi}(\mathbf{x}) - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\left[\mathbf{K}_1(\mathbf{x})\mathbf{x} + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\right] = \\ &\begin{bmatrix} \dot{\mathbf{K}}_1(\mathbf{x}) + \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\mathbf{A} - \\ \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_1(\mathbf{x}) \end{bmatrix}\mathbf{x} + \\ &\begin{bmatrix} \dot{\mathbf{K}}_2(\mathbf{x}) + \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\mathbf{F} - \\ \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_2(\mathbf{x}) \end{bmatrix}\mathbf{\Psi}(\mathbf{x}) \end{aligned} \tag{15}$$

To be consistent with [4,11], to perform optimality criterion, the equation (15) may be written in form where the optimal control for (1) and (2) may be found by solving nonlinear equation system with unknown $\mathbf{K}_1(\mathbf{x})$, $\mathbf{K}_2(\mathbf{x})$ related to linear and nonlinear state $\mathbf{x}$ and $\mathbf{\Psi}(\mathbf{x})$.

$$\begin{aligned} &\begin{bmatrix} \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\mathbf{A} - \\ \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_1(\mathbf{x}) \end{bmatrix}\mathbf{x} + \\ &\begin{bmatrix} \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\mathbf{F} - \\ \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_2(\mathbf{x}) \end{bmatrix}\mathbf{\Psi}(\mathbf{x}) + \\ &\left[\left[\mathbf{A}^T + \mathbf{J}^T(\mathbf{x})\mathbf{F}^T\right]\mathbf{K}_1(\mathbf{x}) + \mathbf{Q}\right]\mathbf{x} + \\ &\left[\mathbf{A}^T + \mathbf{J}(\mathbf{x})^T\mathbf{F}^T\right]\mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x}) = \mathbf{0} \end{aligned} \tag{16}$$

The equation we may separate considering terms related to $\mathbf{x}$ and $\mathbf{\Psi}(\mathbf{x})$. Then, we have two nonlinear equations with unknown $\mathbf{K}_1(\mathbf{x})$, $\mathbf{K}_2(\mathbf{x})$:

$$\begin{aligned} &\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\left[\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_1(\mathbf{x})\right] + \\ &\left[\mathbf{A}^T + \mathbf{J}(\mathbf{x})^T\mathbf{F}^T\right]\mathbf{K}_1(\mathbf{x}) + \mathbf{Q} = \mathbf{0} \end{aligned}, \tag{17}$$

$$\begin{aligned} &\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{J}(\mathbf{x})\right]\left[\mathbf{F} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}_2(\mathbf{x})\right] + \\ &\left[\mathbf{A}^T + \mathbf{J}^T(\mathbf{x})\mathbf{F}^T\right]\mathbf{K}_2(\mathbf{x}) = \mathbf{0} \end{aligned}. \tag{18}$$

The system of equation (17)-(18) is presented in more suitable form by (12)-(13).

Proof From (9) we have

$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}^* \\ \mathbf{p}^*\end{bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\left[\mathbf{A}^T + \mathbf{J}^T(\mathbf{x})\mathbf{F}^T\right]\end{bmatrix}\begin{bmatrix}\mathbf{x}^* \\ \mathbf{p}^*\end{bmatrix} + \begin{bmatrix}\mathbf{F}\mathbf{\Psi}(\mathbf{x}) \\ \mathbf{0}\end{bmatrix}, \tag{19}$$

furthermore vectors $\mathbf{x}^*$, $\mathbf{p}^*$ satisfy $\mathbf{x}^*(t_0) = \mathbf{x}_0$, $\mathbf{\Psi}(\mathbf{x}_0) = \mathbf{\Psi}_0$, $\mathbf{p}^*(\infty) = 0$ and $\mathbf{x}^*(\infty) = 0$. So the pair $\mathbf{x}^*$, $\mathbf{p}^*$ satisfy (19).

The uniqueness can be shown as follows. If ($\mathbf{x}_1, \mathbf{p}_1$) and ($\mathbf{x}_2, \mathbf{p}_2$) satisfy (19) for $\mathbf{x}_1$ close to $\mathbf{x}_2$ and $\mathbf{x}$ lies somewhere between $\mathbf{x}_1$ and $\mathbf{x}_2$, then $\Delta\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ and $\Delta\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$ satisfy

$$\frac{d}{dt}\begin{bmatrix}\Delta\mathbf{x} \\ \Delta\mathbf{p}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\left[\mathbf{A}^T + \mathbf{J}^T(\mathbf{x})\mathbf{F}^T\right]\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x} \\ \Delta\mathbf{p}\end{bmatrix} + \begin{bmatrix}\mathbf{F}\mathbf{\Psi}(\mathbf{x}) \\ \mathbf{0}\end{bmatrix} \tag{20}$$

for $t \in [t_0, \infty]$, $\Delta\mathbf{x}(t_0) = 0$ and $\Delta\mathbf{p}(\infty) = 0$. It implies that