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Standard uncertainty evaluation of multivariate polynomial

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ABSTRACT

The uncertainty propagation law helps to infer the uncertainty of unobservable variables from known or assumed relationship with observable variable. Currently only analytical linear approximation or Monte Carlo simulation methods is widely adopted. The former method is limited to weakly nonlinear systems while the latter does not provide any analytical expression that links the *uncertainties of input* to *uncertainties of output* quantities. This paper proposes procedures to evaluate the standard uncertainty of multivariate polynomial using basic algebraic manipulation and tabulated Mellin transform. Case studies are presented whereby the effectiveness and practicality of the proposed method is demonstrated. The proposed method can be readily automated using computer algebra systems, thus negating the need for practitioners to perform the actual complex computation. The work theoretically enriches and extends the validity of the analytic approach in the existing uncertainty evaluation framework, thus enabling the analytic evaluation of uncertainty for many nonlinear cases which was previously an impossible task.

1. Introduction

The field of measurement often involves indirect determination of an unobservable quantity Y through the measurement of other observable quantities X_i. Assuming that the relationship between input and output quantities $Y = g(X_i,\ldots,X_N)$ is known, the uncertainty of Y depends functionally on the measurement of X_i. The uncertainty is typically evaluated using *The Guide to the Expression of Uncertainty in Measurement* (GUM) [1] which is the most widely referenced document for the evaluation of measurement uncertainty [2]. The GUM uncertainty evaluation framework can be generally partitioned into two major themes: the first estimates the uncertainty of directly observable variables; the second propagates the uncertainties of input quantities to estimate the uncertainty of output quantities. This paper focuses on the second theme with the intention to extend the GUM framework on uncertainty propagation to arbitrary polynomial functions.

The GUM framework introduces an uncertainty propagation law which helps to infer the uncertainty of unobservable variables from known or assumed relationship with observable ones. This is indispensible in many applications where the quantity of interest is a conceptual construction that cannot be directly measured, such as power efficiency, calibration accuracy and rate of chemical reaction. The GUM uncertainty evaluation framework outlines three methods to evaluate the propagation of uncertainty:

a) Analytical methods using mathematical representation of the probability density function (pdf).
b) Uncertainty propagation based on analytic linear approximation (first-order Taylor series expansion) of the model. This is also called the law of uncertainty propagation: $U[y] = \sqrt{\sum_i \left[\frac{\partial g(X_i)}{\partial X_i} U[x]\right]^2}$ where U [·] denotes the standard uncertainty.
c) *Monte Carlo* (MC) Simulation.

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Fig. 1(a) outlines the procedure of applying the analytic first-order Taylor series approximation of a function also known as the *GUM uncertainty propagation framework*. All expected values of variables X_i must be estimated as μ_i. The sensitivity coefficient is the result of implementing a local linear approximation, which is the partial derivative of the function $g(\cdot)$ evaluated at the point $X_i = \mu_i$. The standard uncertainty $U[Y]$ can then be calculated from known uncertainties of input variables $U[X_i]$ and the corresponding sensitivity coefficients. The analytical expression provides insight into the relationship between uncertainties of the output variable and uncertainties of various input quantities. However due to its reliance on the local linear approximation procedure, this method will be inaccurate for highly nonlinear models.

Recognizing this methodological gap, the MC simulation framework shown in Fig. 1(b) was presented in the GUM Supplement 1 [3]. The MC method simulates many independent realizations of the input quantities using its *a priori* probability distribution. The distribution is then propagated to the output by directly evaluating the input quantities. Thus, the *posterior* distribution of the output quantity can be reliably estimated given that a sufficient number of simulations or trials are used. The standard uncertainty of the output variable is directly calculated from the *posterior* distribution. This is known as the propagation of distribution which is the most general form of the uncertainty propagation law. MC simulation method is a very powerful approach which is valid for wider classes of uncertainty estimation problems compared to the GUM uncertainty propagation framework.

MC requires that sufficiently high number of realizations is needed to achieve good probability convergence. The number of realizations needed is dictated by the required probability coverage and the type of distribution under consideration. Unlike the analytical method, the MC simulation method will not provide any analytical expression that links the uncertainties of input to output quantities. Thus the relationship between uncertainties of the output variable and uncertainties of various input quantities remains opaque and may only be determined through substantial additional modelling efforts.

The analytic solution is important because the explicit expression of how the output variable uncertainty is

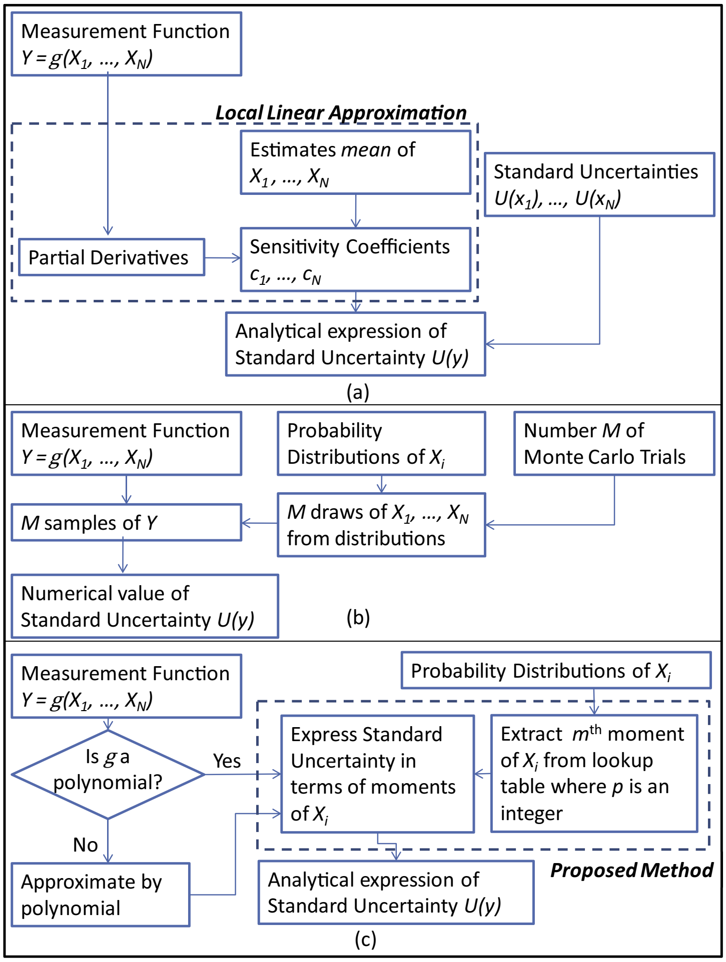

Fig. 1. (a) The GUM uncertainty propagation framework [1]; (b) The GUM Supplement 1 on MC simulation for uncertainty propagation [3]; (c) Proposed multivariate polynomial uncertainty propagation framework.