Available online at www.sciencedirect.com
ScienceDirect
IFAC-PapersOnLine 48-11 (2015) 434–439

# Robust Quadratic Stabilization of Bilinear Control Systems

**Mikhail V. Khlebnikov** *

* *Institute of Control Science, Profsoyuznaya 65, Moscow, Russia (e-mail: khlebnik@ipu.ru)*

**Abstract:** In this talk, the problem of robust stabilization of bilinear control systems is considered. Based on the technique of linear matrix inequalities and quadratic Lyapunov function, the so-called robust stabilizability ellipsoid is designed such that the trajectory of the closed-loop system, starting inside the ellipsoid asymptotically tends to zero for all admissible uncertainties. The results obtained allow to design an approximation of the robust stabilizability domain of the uncertain bilinear control system.



*Keywords:* bilinear control system, linear static state feedback, robustness, quadratic stabilization, linear matrix inequalities, stabilizability ellipsoid, stabilizability domain

## 1. INTRODUCTION

Problems of stabilization of bilinear control systems has got a lot of attention in the literature, especially since the appearance of the monograph Mohler (1973); see also Ryan and Buckingham (1983); Chen et al. (1991), and others. At the same time, a variety of different approaches was proposed. For example, in Čelikovský (1990, 1993), linear transformation is searched such that the bilinear system becomes a linear one. For other areas associated with using the observers, see Tibken et al. (1996).

Note also the works by Belozyorov (2002, 2005), where, on the basis of sufficient conditions of stability for quadratic systems of differential equations, a method of constructing a linear control for a bilinear systems is developed. In the certain papers the ellipsoidal approach is used to the topic, see e.g. Tibken et al. (1996). We note also a number of works devoted to the design of nonlinear control laws for stabilization of bilinear systems, e.g., see Coutinho and de Souza (2012); Andrieu and Tarbouriech (2013); Omran et al. (2014); Kung et al. (2012).

Some of the most ideologically close works by Amato et al. (2009); Tarbouriech et al. (2009), are devoted to the approaches based on the technique of linear matrix inequalities (LMI) for the construction of quadratic Lyapunov functions for stabilization of bilinear control systems.

The present paper however differs a lot; above all, it deals with an uncertain bilinear control system. Further, on top of a somewhat different statement of the problem, we formulate the problem of maximizing stabilizability ellipsoids with respect to a certain criterion. Moreover, in contrast to the above mentioned papers, a new problem of constructing stabilizability domains is formulated and solved. Finally, we use a different technique based on the modification of Petersen lemma.

Namely, based on the LMI-technique, see Boyd et al. (1994), and special modification of the Petersen lemma we propose a regular approach to the robust stabilization of bilinear control systems via a linear static state feedback. In the state space of the system, we design an ellipsoid (so-called robust stabilizability ellipsoid), such that the trajectory of the closed-loop system, starting inside the ellipsoid asymptotically tends to zero for all admissible system uncertainties. The natural development of this approach allows to effectively design an approximation of the robust stabilizability domain of bilinear control systems.

We stress that the proposed approach is based on the solution of convex optimization problems, yet allows for the construction of nonconvex approximations of robust stabilizability domains of uncertain bilinear systems.

The MATLAB-based toolbox `cvx`, see Grant and Boyd (2014), was used for computations.

## 2. UNCERTAINTY-FREE CASE

### 2.1 Statement of the Problem

Consider the bilinear control system

$$\dot{x} = Ax + bu + Dxu, \quad x(0) = x_0, \tag{1}$$

with state $x \in \mathbb{R}^n$ and scalar control input $u \in \mathbb{R}$; here $A, D \in \mathbb{R}^{n\times n}$, and $b \in \mathbb{R}^n$.

We will design a linear static state feedback

$$u = k^\top x, \quad k \in \mathbb{R}^n, \tag{2}$$

which quadratically stabilizes system (1) inside a certain ellipsoid

$$\mathcal{E} = \left\{x \in \mathbb{R}^n : \quad x^\top P^{-1} x \le 1\right\} \tag{3}$$

with matrix $P \succ 0$ and center at the origin. In other words, the trajectories of bilinear system (1) under control input (2), starting at any point $x_0$ inside the ellipsoid $\mathcal{E}$ asymptotically tend to zero.

We call the ellipsoid $\mathcal{E}$ a *stabilizability ellipsoid* (*SE*) for bilinear system (1) corresponding to control (2). Further we will try to enlarge in some sense the SE.

2405-8963 © 2015, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.09.224

*2.2 Main Result*

The so-called Petersen lemma, see Petersen (1987), is effectively used in various robust statements of the problems of stabilization and control. We present it in the following formulation. Here and below, $I$ is the unit matrix of appropriate dimension, all matrix inequalities are understood in the sense of matrix sign-definiteness, and $\|\cdot\|$ denotes the spectral matrix norm.

*Lemma 1.* Let $G = G^\top \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{q\times n}$. The inequality

$$G + M\Delta N + N^\top \Delta^\top M^\top \prec 0$$

is valid for all $\Delta \in \mathbb{R}^{p\times q}$: $\|\Delta\| \le 1$ if and only if there exists a number $\varepsilon$ such that

$$\begin{pmatrix} G + \varepsilon M M^\top & N^\top \\ N & -\varepsilon I \end{pmatrix} \prec 0. \qquad \blacktriangledown$$

Some generalizations of the Petersen lemma was cosidered in Khlebnikov and Shcherbakov (2008); Khlebnikov (2014), as well as its simple proof on the basis of $S$-procedure, see Yakubovich (1973). Here we give the following modification of the Petersen lemma concerned with the case of vector uncertainty subjected to the ellipsoidal constraint.

*Lemma 2.* Let $G = G^\top \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times q}$, $N \in \mathbb{R}^{1\times n}$, and $0 \prec P = P^\top \in \mathbb{R}^{q\times q}$. The inequality

$$G + M\delta N + N^\top \delta^\top M^\top \prec 0$$

is valid for all $\delta \in \mathbb{R}^q$: $\delta^\top P^{-1}\delta \le 1$ if and only if there exists a number $\varepsilon$ such that

$$\begin{pmatrix} G + \varepsilon M P M^\top & N^\top \\ N & -\varepsilon I \end{pmatrix} \prec 0. \qquad \blacktriangledown$$

Therefore, the verification for sign-definiteness of the family $G + M\delta N + N^\top \delta^\top M^\top$ is reduced by Lemma 2 to the problem of solvability of LMI with respect to one scalar variable $\varepsilon$. In the sequel, this result will be used in the most significant way.

Now we establish the following main result.

*Theorem 3.* Let the matrix $P$ and the vector $y$ satisfy the matrix inequalities

$$\begin{pmatrix} AP + PA^\top + by^\top + yb^\top + \varepsilon DPD^\top & y \\ y^\top & -\varepsilon I \end{pmatrix} \prec 0,$$

$$P \succ 0,$$

for a certain value of the scalar parameter $\varepsilon$.

Then the linear static state feedback (2) with the gain matrix

$$k = P^{-1} y$$

stabilizes system (1) inside the ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^n : \quad x^\top P^{-1} x \le 1\}.$$

Moreover, the quadratic form

$$V(x) = x^\top P^{-1} x$$

is a Lyapunov function for the closed-loop system (4) inside the ellipsoid $\mathcal{E}$. $\blacktriangledown$

**Proof.** Embracing system (1) with feedback (2), we obtain the closed-loop bilinear system

$$\dot{x} = A_c x + D x k^\top x \tag{4}$$

with matrix $A_c = A + bk^\top$.

Consider the quadratic form

$$V(x) = x^\top Q x, \quad Q \succ 0,$$

and obtain the conditions under which it will be a Lyapunov function for system (4). To this end, we calculate its derivative along the trajectories of the closed-loop system:

$$\dot{V}(x) = (A_c x + D x k^\top x)^\top Q x + x^\top Q (A_c x + D x k^\top x) =$$
$$x^\top A_c^\top Q x + x^\top Q A_c x + x^\top Q D x k^\top x + x^\top k x^\top D^\top Q x =$$
$$x^\top (A_c^\top Q + Q A_c + Q D x k^\top + k x^\top D^\top Q) x.$$

Hence, if the condition

$$A_c^\top Q + Q A_c + Q D x k^\top + k x^\top D^\top Q \prec 0$$

holds, the quadratic form $V(x)$ is the Lyapunov function for system (4).

Multiplying the obtained condition from the left and right side by the matrix $P = Q^{-1} \succ 0$ we obtain

$$A_c P + P A_c^\top + D x k^\top P + P k x^\top D^\top \prec 0. \tag{5}$$

We require that the matrix inequality (5) hold for any $x$ inside the ellipsoid (3). Thus, $V(x)$ is the quadratic Lyapunov function for the closed-loop system inside ellipsoid.

By Lemma 2, we obtain the equivalent matrix inequality

$$\begin{pmatrix} A_c P + P A_c^\top + \varepsilon D P D^\top & Pk \\ k^\top P & -\varepsilon I \end{pmatrix} \prec 0. \tag{6}$$

Introducing the new vector variable

$$y = Pk,$$

we eliminate $k$. By virtue of $P \succ 0$, the vector $k$ is restored in a unique way as $k = P^{-1} y$. In such way we obtain the following inequality

$$\begin{pmatrix} AP + PA^\top + by^\top + yb^\top + \varepsilon DPD^\top & y \\ y^\top & -\varepsilon I \end{pmatrix} \prec 0$$

with the scalar parameter $\varepsilon$, which is linear with respect to the matrix variable $P$ and the vector variable $y$. $\blacktriangledown$

Note that the feasibility of the matrix inequality (6) yields the Hurwitz stability of the matrix $A_c$. This means that the required control (2) stabilizes the linear control system $\dot{x} = Ax + bu$, see Polyak et al. (2014).

It is naturally to maximize the SE by a certain criterion; in particular, we maximize the radius of the contained ball.

*Corollary 4.* Let $\widehat{P}$ and $\widehat{y}$ be the solution of the convex optimization problem

$$\max \lambda_{\min}(P) \tag{7}$$

subject to the constraints

$$\begin{pmatrix} AP + PA^\top + by^\top + yb^\top + \varepsilon DPD^\top & y \\ y^\top & -\varepsilon I \end{pmatrix} \prec 0,$$

$$P \succ 0,$$

with respect to the matrix variable $P = P^\top \in \mathbb{R}^{n\times n}$, the vector variable $y \in \mathbb{R}^n$, and the scalar parameter $\varepsilon$.

Then

$$\widehat{\mathcal{E}} = \{x \in \mathbb{R}^n : \quad x^\top \widehat{P}^{-1} x \le 1\}$$

is the SE for bilinear system (1) with feedback (2) defined by the gain matrix

$$\widehat{k} = \widehat{P}^{-1} \widehat{y}. \qquad \blacktriangledown$$