SELECTING PATH FLOW PATTERN FOR SENSITIVITY ANALYSIS OF CAPACITATED NETWORK OF USER EQUILIBRIUM

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Abstract: Sensitivity analysis is a means for extracting the cause and effect relationship between inputs and outputs in the network model. The paper proposes the scheme of integrating column generation and penalization of capacity constraints, to find a set of equilibrium path flow pattern of the capacitated network. The solution of path flows becomes candidate flow patterns, from which we can abstract extreme points specially used for sensitivity analysis, by using a linear programming. Numerical example shows the effect of change in input parameters on flow and delay.

Keywords: sensitivity; network; capacity; constraints; equilibrium; transportation; traffic control; Newton; linear programming.

1. INTRODUCTION

The network flow model attempts to duplicate the vehicular flows on the network, by partitioning the origin-destination trip rates between the observed paths. The sensitivity analysis is a method for extracting the cause and effect relationship between the inputs and outputs in the network flow model. The basic idea is that each input channel to the network is offset slightly and the corresponding change in the output is reported. For example, link time or origin-destination demand is likely to contain uncertainties because of the difficulty in obtaining accurate data; the input channels that produce high sensitivity values can be considered significant and can most often be paid attention in network design and traffic operation.

The sensitivity of the deterministic user equilibrium (UE) was developed as that of a restricted variational inequality (Tobin and Friesz, 1988). The sensitivity in the context of the logit-based stochastic UE was addressed as a link-based approach (Ying and Miyagi, 2001), implemented in the sophisticated procedure of the Dial's algorithm, and the sensitivity in the probit-based stochastic UE was explored on the basis of the information on the stochastic UE path flows (Clark and Watling, 2002). The sensitivity analysis of deterministic or stochastic UE has been applied for solving congestion pricing and network design problems (Yang 1997; Ying, 2005).

The sensitivity of the deterministic UE requires a previous set of reference path flows, from which an extreme path flow pattern might be abstracted. Unfortunately, how to obtain the candidate path flow pattern is not solved. The traffic condition of deterministic UE is unique with link flows, but not unique with respect to path flows. A link flow pattern may correspond to several path flow patterns (Patriksson, 1994; Bar-Gera, 2002). Even when the link flow pattern is known, finding an equilibrium path flow is still not a trivial task. For a small network, all the paths could easily be enumerated as the reference set, but for a large network, path enumeration is unimaginable.

The objective of this paper is to demonstrate a comprehensive methodology for investigating factor sensitivities of deterministic UE model over the capacitated network. We first present how to find a set of equilibrium path flow pattern by using the Newton method, a path-based algorithm, which integrates Newton formula and column generation; second, we illustrate how to abstract the extreme path flow pattern used for flow sensitivity analysis, by using a linear programming. The two problems play a significant role in the applying sensitivity analysis to traffic network practices. A numerical example is finally demonstrated for the deterministic UE models over the capacitated network.

2. PATH-FORMULATED NETWORK MODEL

Given a transport network $G(A, N)$, where A and N are the sets of links and nodes respectively, and each directed link $a \in A$ is associated with an increasing travel time $t_a(x_a)$ as an function of link flow x_a. W is the set of origin destination pairs, and for each pair $w \in W$, there is a given traffic demand q^w. The user equilibrium assignment problem in the capacitated network is formulated as follows:

$$\text{minimize } z(\mathbf{x}) = \sum_a \int_0^{x_a} t_a(x) \cdot dx \qquad (1a)$$

subject to

$$\sum_k f_k^w = q^w \qquad w \in W \text{ (or } \mathbf{\Lambda f} = \mathbf{q}) \qquad (1b)$$

$$x_a = \sum_w \sum_k f_k^w \cdot \delta_{ak}^w \qquad \forall a \in A \text{ (or } \mathbf{x} = \mathbf{Af}) \qquad (1c)$$

$$x_a \le c_a \qquad (d_a) \quad \forall a \in A \qquad (1d)$$

$$f_k^w \ge 0 \qquad \forall k \in K^w, w \in W \qquad (1e)$$

where f_k^w denotes the flow on path k within OD pair w, $\mathbf{f}=[f_k^w]$, and K^w is the path set within OD pair w. ca is the capacity of link a. d_a in the bracket is the Lagrange multiplier of the corresponding equation $(1d)$. d_a is positive if the equation is active and zero otherwise. δ_{ak}^w =1 if link is in the path k; otherwise, δ_{ak}^w =0, $\mathbf{A} = [\delta_{ak}^w]$.

This problem becomes conventional UE assignment if dropping the link capacity constraints $(1d)$. Since all path flow variables of interest are positive and nonnegativity constraints $(1e)$ are not binding, the nonnegativity in terms of path flows may be omitted in the following without affecting the assignment solution. The path set will only include those positive variables or say used paths through all the paper. The following Lagrange function will be used to build the necessary equation at optimality.

$$L[\,\cdot\,] = \sum_a \int_0^{x_a} t_a(x) + \mathbf{u}^T(\mathbf{q} - \mathbf{\Lambda f}) + \mathbf{d}^T(\mathbf{Af} - \mathbf{c}) \qquad (2)$$

In terms of path flows, the derivative of the Lagrange function equals zero, so that,

$$\boldsymbol{\tau} - \mathbf{\Lambda}^T\mathbf{u} + \mathbf{A}^T\mathbf{d} = \mathbf{0} \qquad (3)$$

Where, $\mathbf{u}$ is the vector of Lagrange multiplier with elements, $u^w = \min_k\{t_a \cdot \delta_{ak}^w\}$, the shortest path time within OD pair w. τ is the time vector corresponding to the used paths, with elements, $\tau_k^w = \Sigma_a t_a \cdot \delta_{ak}^w$. $\boldsymbol{d}$ is the vector of the Lagrange multiplier of the equation $(1d)$, $\boldsymbol{d}=[d_a]$.

The solution of the capacitated network model has the characterization of a Wardrop principle when the travel time is articulated in terms of running time and waiting time. This generalized travel time is, in fact, the ordinary one to be minimized by the individual travellers in a congested network. The waiting time or queuing delay is equivalent to the Lagrange multiplier associated with the capacity constraint on the given link. Then equilibrium flow pattern and generalized time over the capacitated network can be obtained once the problem $(1a)$-$(1e)$ is solved

3. FINDING PATH FLOW PATTERN

The path-based method is adopted for solving the path-formulated network flow model and finding the candidate path flow pattern. Column generation and line search are incorporated into the solution procedure. The column generation allows that the active paths joining each OD pair are generated endogenously only when needed. The convex line search is implemented to avoid the infeasibility of flows on the used paths and equilibrate the demand among the active paths as well as the newly founded shortest path for every OD pair. The Newton method (Cheng et al, 2003) is reported to be superior to the gradient projection (Bertsekas and Gallager, 1987) in the robustness of convergence.

3.1 Ideas of Newton Method in Path Flow Domain

Different from the optimization of the link-based algorithm, which is performed in the link-flow domain, the Newton method is performed in the path-flow domain. With respect to path flow vector only inclusive of used elements, the proposed method iteratively solves problem (1) from a feasible point to an improved feasible point. Provided a feasible point or say path flow vector f n, a moving flow $\Delta\mathbf{f}$ n and a the step size λ are determined such that the following two properties are true:

- $\mathbf{f}^n + \lambda \cdot \Delta\mathbf{f}^n$ is feasible
- $\mathbf{f}^n + \lambda \cdot \Delta\mathbf{f}^n$ is better than that at $\mathbf{f}^n$

This leads to a new point

$$\mathbf{f}^{n+1} = \mathbf{f}^n + \lambda \cdot \Delta\mathbf{f}^n \qquad (4)$$

The process above mentioned is repeated until the objective function cannot be further improved. In the space of path flows, the objective function $z(\mathbf{x})$ in equation $(1a)$ is approximated by the following second-order Taylor series:

$$\begin{aligned} z(\mathbf{f}) &\cong z(\mathbf{f}^n) + (\mathbf{f} - \mathbf{f}^n)^T \cdot \boldsymbol{\tau}(\mathbf{f}^n) \\ &+ \tfrac{1}{2}(\mathbf{f} - \mathbf{f}^n)^T \cdot \boldsymbol{\tau}'(\mathbf{f}^n) \cdot (\mathbf{f} - \mathbf{f}^n) \end{aligned} \qquad (5)$$

where $\tau(\mathbf{f}^n)$, $\tau'(\mathbf{f}^n)$ denote the vectors of path costs and their derivatives, which are related to the first and second order derivatives of the objective function to the respective path flows. $\mathbf{f}^n$ is the vector of current path flow solutions. The minimizing is performed at the approximate function. The first derivative of the approximate function is:

$$\begin{aligned} \boldsymbol{\tau}(\mathbf{f}) &= \boldsymbol{\tau}(\mathbf{f}^n) + (\mathbf{f} - \mathbf{f}^n)^T \cdot \boldsymbol{\tau}'(\mathbf{f}^n) \\ &= \boldsymbol{\tau}(\mathbf{f}^n) + \Delta\mathbf{f}^n \cdot \boldsymbol{\tau}'(\mathbf{f}^n) \end{aligned} \qquad (6)$$