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Optimal State Estimation for Linear Systems with State Constraints

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Abstract: This paper considers the optimal strategies for constrained linear state estimation. Prior information for estimating state variables is often available in the form of inequality constraints on states. In the latest developments of optimal state estimation theory consideration of state constraints has been often neglected since constraints do not fit easily in the structure of the optimal filter, for example, the issue of state constraints being present has to be addressed adequately, for example nonnegativity of concentration. In order to address this issue and to extend previous developments on the accuracy of state estimation, this work develops the constrained optimal state estimation for finite-dimensional systems. Finally, a numerical example illustrating the proposed method is presented.

Keywords: Optimal state estimation, Inequality constraints, Linear time-invariant system.

1. INTRODUCTION

In finite-dimensional systems, the Kalman filter is the standard choice for estimating the state of a linear system when the measurements are noisy and the process disturbances are unmeasured. Another important estimation technique, given by moving horizon method which is developed by Muske and Rawlings (1993), plays an important role in finite-dimensional systems estimation. Often in the practice, the additional information for estimation is available in the form of inequality constraints on states. In order to embed the constraints into the state estimation framework and improve the accuracy of state estimation, many contributions have been made (see Muske et al. (1993), Simon and Chia (2002), Simon (2010), and Rao et al. (2001)).

Thomas et al. Yu et al. (1974) investigated the optimal state estimation framework for infinite-dimensional systems by utilizing the framework of the optimal control theory. Along the same time, Ray (1981) summarized and applied this framework in both lumped parameter and distributed parameter systems (see Ajinkya et al. (1975); Soliman and Ray (1979)). The optimal state estimation technique developed by Thomas and Ray was formulated by utilizing variational method for continuous systems and the resulting estimation formulations are continuous time functions. In this paper, motivated by Ray (1981), we extended the framework of continuous optimal state estimation technique to deal with the state estimation problem when the state constraints for continuous linear time-invariant system are explicitly included.

Motivated by consideration above, this paper focuses on the development of the constrained optimal state estimation framework. Contrary to the Kalman filtering and moving horizon techniques, this paper deals with the optimal constrained state estimation problem based on the continuous systems using the variational method and the final state estimation formulation which is given as a continuous time function. The contribution of this paper is its novel methodology which first converts the optimal state estimation problem with inequality constraints into the problem with equality constraints and embeds the equality constraints in the optimal state estimation framework. The derivation of a state estimation formulation is demonstrated in Section 2. In Section 3, a numerical example, via simulation, shows that the proposed state estimation framework in this paper indeed improves the accuracy of state estimation compared with the unconstrained optimal state estimation framework (Ray's method). Finally, the conclusion is presented in Section 4.

2. MODEL DESCRIPTION

Let us consider the following linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t) + G\xi(t),\ x(0) = x_0 \tag{1}$$

$$y(t) = Cx(t) + \eta(t) \tag{2}$$

where $x(t) \in R^n$, $u(t) \in R$, $y(t) \in R$ are state, input and output, respectively and $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $G \in R^{n\times 1}$, and $C \in R^{1\times n}$ are state, input, disturbance and output matrices, respectively. $\xi(t)$ and $\eta(t)$ are the zero-mean random processes with the following stochastic properties:

$$\begin{aligned} &E(\xi(t)) = 0, E(\xi(t)\xi(\tau)^T) = R^{-1}(t)\delta(t-\tau)\\ &E(\eta(t)) = 0, E(\eta(t)\eta(\tau)^T) = Q^{-1}(t)\delta(t-\tau)\\ &E(\xi(t)\eta(\tau)^T) = 0 \end{aligned} \tag{3}$$

The state $x(t)$ in the system (1) is subjected to the following constraint:

$$\mathcal{X}^{\min} \le \Gamma x(t) \le \mathcal{X}^{\max} \tag{4}$$

where $\Gamma \in R^{1\times n}$ is a vector.

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2.1 State Estimation Formulation

In this section, based on the system (1)-(2), we formulate the constrained optimal state estimation problem as the solution to the following quadratic problem:

$$\min_{\hat{x}(t)} \quad J(\hat{x}(t)) \tag{5}$$

where the objective function is defined by:

$$\begin{aligned} J &= \frac{1}{2}[\hat{x}(0) - x_0]^T P_0^{-1}[\hat{x}(0) - x_0] \\ &+ \frac{1}{2}\int_0^{t_f} \left\{ \left(\dot{\hat{x}}(t) - A\hat{x}(t) - Bu(t)\right)^T \right. \\ &\left. \times G^T RG \left(\dot{\hat{x}}(t) - A\hat{x}(t) - Bu(t)\right)\right\} dt \\ &+ \frac{1}{2}\int_0^{t_f} \left\{ (y(t) - C\hat{x}(t))^T Q \ (y(t) - C\hat{x}(t))\right\} d \end{aligned}$$

subject to constraint:

$$\mathcal{X}^{\min} \le \Gamma\hat{x}(t) \le \mathcal{X}^{\max} \tag{6}$$

where (6) is consistent with (4), t_f is terminal time, $\hat{x}(t)$ is the estimation of the state $x(t)$, and $R(t)$, $Q(t)$ are chosen in (3) and P_0 is defined by:

$$E([\hat{x}(0) - x_0]\,[\hat{x}(0) - x_0]^T) = P_0 \tag{7}$$

We shall now define $U(t) = \dot{\hat{x}}(t) - A\hat{x}(t) - Bu(t)$ to convert the optimal state estimation problem to its dual optimal control problem:

$$\min_{U(t)} \quad J(\hat{x}(t)) \tag{8}$$

where the objective function is defined by:

$$\begin{aligned} J &= \frac{1}{2}[\hat{x}(0) - x_0]^T P_0^{-1}[\hat{x}(0) - x_0] \\ &+ \frac{1}{2}\int_0^{t_f} \left\{ U(t)^T G^T RG\, U(t)\right\} dt \\ &+ \frac{1}{2}\int_0^{t_f} \left\{ (y(t) - C\hat{x}(t))^T Q \ (y(t) - C\hat{x}(t))\right\} dt \end{aligned}$$

subject to constraints:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + U(t), \ \hat{x}(0) \text{ unspecified} \tag{9}$$

$$\mathcal{X}^{\min} \le \Gamma\hat{x}(t) \le \mathcal{X}^{\max} \tag{10}$$

One may solve the optimization problem (8), (9) and (10) by realizing the following two algorithmic steps:

P.1) One solves the optimization problem (8) and (9) without the constraint (10). Then, inspect if the results satisfy the constraint given by (10). If the results satisfy the constraint, then one finishes the estimation work at the current estimation time instant. Otherwise, we proceed to step (P.2), in other words, the constraint is not active.

P.2) In this step, one inspects which part of the constraint (10) is not satisfied. In the case that the estimation results do not satisfy $\mathcal{X}^{\min} < \Gamma\hat{x}(t)$, one needs to resolve the optimization problem (8) and (9) subject to $\mathcal{X}^{\min} < \Gamma\hat{x}(t)$. According to section 11.2.2 of Simon (2006), in this step, the inequality optimization problem is converted into the equality constrained optimization problem:

$$\begin{aligned} &\min J(\hat{x}(t)) \\ &\text{s.t. (9) and } S(\hat{x}, t) = -\Gamma\hat{x}(t) + X^{\min} = 0 \end{aligned} \tag{11}$$

In the same way, if the estimation results do not satisfy $\Gamma\hat{x}(t) < \mathcal{X}^{\max}$, one needs to resolve the problem:

$$\begin{aligned} &\min J(\hat{x}(t)) \\ &\text{s.t. (9) and } S(\hat{x}, t) = -\Gamma\hat{x}(t) + X^{\max} = 0 \end{aligned} \tag{12}$$

1). In step (P.1), we directly formulate the unconstrained state estimator according to Ray (1981) and the formulation will be given at the end of this section.
2). In step (P.2), we embed the inequality constraints within the Ray's optimal state estimation framework. Essentially, the problems (11) and (12) are the same, in this paper we use the problem (11) as representative to illustrate the derivation of formulation and finally we directly give the formulation for the case (12).

According to Bryson (1975), it is easier to deal with the equality constrained optimal control problems through variational method when the constrained function contains an explicit expression of the control variable i.e. $U(t)$, which is the case in this paper.
Consider the following constraint:

$$S(\hat{x}, t) = -\Gamma\hat{x}(t) + \mathcal{X}^{\min} = 0 \tag{13}$$

Since $S(\hat{x}, t)$ does not contain the explicit expression of $U(t)$, an additional formulation needs to be developed. If the constraint (13) is applied for all $0 \le t \le t_f$, its time derivative along the path must vanish, i.e.,

$$\frac{dS(\hat{x}, t)}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial \hat{x}}\dot{\hat{x}} = 0 \tag{14}$$

Substituting (9) into (14), one obtains:

$$\Gamma A\hat{x}(t) + \Gamma Bu(t) + \Gamma U(t) = 0 \tag{15}$$

Apparently, (15) has explicit dependence on $U(t)$ and thus plays the role of a control variable constraint of the type (3.3.1) in Bryson (1975). In this case, we formulate the minimization problem as:

$$\begin{aligned} &\min J(\hat{x}(t)) \\ &\text{s.t. (9) and } \Gamma A\hat{x}(t) + \Gamma Bu(t) + \Gamma U(t) = 0 \end{aligned} \tag{16}$$

We first formulate the augmented Hamiltonian:

$$\begin{aligned} H = &\frac{1}{2}U^T(t)R_G U(t) + \frac{1}{2}(y(t) - C\hat{x}(t))^T Q\,(y(t) - C\hat{x}(t)) \\ &+ \lambda^T(t)\,[A\hat{x}(t) + Bu(t) + U(t)] \\ &- \mu(t)\,[\Gamma A\hat{x}(t) + \Gamma Bu(t) + \Gamma U(t)] \end{aligned}$$

where $R_G = G^T RG$, λ is a Lagrange multiplier vector and μ is a Lagrange multiplier scalar. The last term of Hamiltonian originates from the (15).

In order to guarantee the solvability of the constrained minimization problem (16), the following three conditions have to be satisfied: