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Improved Design of Nonlinear Model Predictive Controllers

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Abstract: One way to ensure recursive feasibility, stability and performance of Nonlinear Model Predictive Control is the combined use of a terminal region and a terminal cost. However, finding suitable combinations of the terminal cost and terminal region that guarantee closed-loop stability for nonlinear systems is in general challenging. Most existing methods are either based on the linearized system dynamics and a linear feedback, or assume that a control Lyapunov function for the system close to the origin is know. This paper proposes the use of higher order approximations of the optimal feedback and optimal cost of the infinite horizon problem via Al'brekht's Method to determine a suitable terminal region for polynomial systems. To do so, the stability conditions are reformulated in terms of a sum-of-squares problem which is iteratively used to determine the terminal region. For a nonlinear chemical reactor example it is shown that the proposed approach leads to a larger terminal region and an improved performance compared to existing approaches.

1. INTRODUCTION

Model Predictive Control (MPC) has achieved a significant success as an advanced control technique especially in its linear version [Rawlings and Mayne, 2009, Mayne, 2014] and also in industrial applications [Qin and Badgwell, 2003]. Despite many results presented in the research community for the nonlinear case, its practical application is still in its early stages. Besides the computational cost associated to solving a nonlinear programming problem and the difficulty of obtaining an accurate nonlinear model of the system, one of the main obstacles for the design of Nonlinear Model Predictive Control (NMPC) is the design of stabilizing controllers that guarantee stability and constraint satisfaction. By now various approaches to guarantee nominal and robust stability of NMPC exist, see e.g. [Rawlings and Mayne, 2009, Mayne, 2014, Findeisen and Allgöwer, 2002, Grüne and Pannek, 2011]. One classical approach is the use additional stabilizing *ingredients* (a terminal cost and terminal constraint) that are included in the optimization problem to achieve a stability guarantee for the closed loop system and recursive feasibility of the optimization problem, see e. g. [Chen and Allgöwer, 1998, Mayne et al., 2000, Grüne and Pannek, 2011, Findeisen and Allgöwer, 2002, Fontes, 2001].

The classical approaches to calculate such ingredients are based on the linearization of the nonlinear dynamics around an equilibrium point and on the computation of a linear feedback control that locally stabilizes the system [Chen and Allgöwer, 1998, Michalska and Mayne, 1993]. This linear feedback is used to compute a quadratic terminal cost and the terminal set is considered to be a sub-level set of this cost. There exist other methods that modify the terminal penalty term in such a way that the terminal constraint is not necessary as shown in [Limon et al., 2006], or in [Grüne, 2012]. In the present paper, we focus on the most used method that employs both a terminal penalty and a terminal constraint.

Several methods presented in the literature use linear differential inclusions (LDI) [Chen et al., 2003, Yu et al., 2009] to approximate the nonlinear system as a linear time-varying system. This method achieves in general invariant regions of a larger volume compared to the ones based on linearization and Lipschitz bounds, e. g. [Chen and Allgöwer, 1998] but leads typically to conservative results because of the LDI representation of the nonlinear system. Further improvements were presented in [Cannon et al., 2003, Barjas Blanco and De Moor, 2007], where polytopic terminal regions are considered and in [Ong et al., 2006], where support vector machines are employed to enlarge the terminal region. However, they share two main difficulties. All of them (with the exception of the work presented in Bacic et al. [2002] for bilinear systems) assume a linear controller of the nonlinear dynamics, based on which the terminal region is computed. This clearly limits the size of the obtained terminal region in the nonlinear case. Additionally, it is usually guaranteed that there exists a sub-level set of the obtained terminal cost which can be suitable terminal region, but it is very difficult to compute the sub-level set with the largest volume, although some iterative schemes such as the one in [Cannon et al., 2003] exist.

In this paper, we facilitate the design of NMPC controllers by alleviating the difficulties mentioned above. We make use of Al'brekht's method [Al'brekht, 1961, Aguilar and Krener, 2014, Krener, 2014, Hunt and Krener, 2010] to calculate a higher degree approximation of the optimal infinite horizon cost and feedback in an efficient manner. We use both the (approximated) optimal cost and optimal feedback, which are polynomials of a degree of choice, to calculate a terminal region that satisfies the necessary conditions of stability for the NMPC

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algorithm. We compute in an efficient manner a possible terminal region for polynomial systems by reducing the problem to a sum-of-squares problem using Putinar's positivstellensatz [Putinar, 1993].

As opposed to most of the literature, we do not use a linear feedback control, but a higher order approximation of it, as well as a high oder approximation of the terminal cost. This, together with the efficient calculation of the terminal set, provides a design method that can achieve a larger terminal region and also increases its performance because the terminal penalty is a better approximation of the infinite horizon cost. These results are illustrated using a nonlinear chemical reactor example.

The remainder of the paper is structured as follows. We describe the problem setup in Section 2, and described the Al'brekht's method in Section 3. Section 4 shows the proposed method to compute the terminal set for polynomial systems. The results for a nonlinear chemical reactor are included in Section 5 and Section 6 concludes the paper.

2. PROBLEM SETUP

We consider continuous-time time-invariant nonlinear systems described by:

$$\dot{x}(t) = f(x(t), u(t)), \quad t \geq 0, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the vector of states, $u \in \mathbb{R}^m$ is the vector of control inputs and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are the dynamics of the nonlinear system. The control task is to locally stabilize an equilibrium point, which for simplicity and without loss of generality is assumed to be at the origin ($f(0, 0) = 0$).

As it is usually done in MPC, we consider a sampled feedback law, which is piecewise constant between sampling times t_i:

$$u(t) = \kappa(x(t)), \quad \forall t \in [t_i, t_{i+1}] \tag{2}$$

At each sampling instant t_i the following optimal control problem is solved:

$$\begin{aligned}
\underset{u(\cdot)}{\text{minimize}} \quad & \int_{t_i}^{t_i+T_p} L(x(\tau), u(\tau))d\tau + E(x(t_i + T_p)) && \text{(3a)}\\
\text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)), \quad x(t_i) = x_{t_i} && \text{(3b)}\\
& x(t_i + T_p) \in \mathbb{X}_f, && \text{(3c)}\\
& x(t) \in \mathbb{X}, u(t) \in \mathbb{U}, && \text{(3d)}\\
& \text{for } t \in [t_i, t_i + T_p],
\end{aligned}$$

where T_p is the prediction horizon. $L(\cdot)$ and $E(\cdot)$ are the stage cost and terminal cost, respectively. The state at the final time in the prediction is constrained to lie in a terminal set $\mathbb{X}_f$ and x_{t_i} denotes the current measurement of the state. $\mathbb{X}$ and $\mathbb{U}$ denote state and input constraints and are assumed to be compact sets that contain the origin. We assume throughout the paper that the dynamics f, the stage cost L and the terminal cost E are C^∞.

After solving (3), the first control input $u_0 = u(t), \forall t \in [t_i, t_{i+1}]$ is applied until the next sampling time in a receding horizon fashion. The resulting closed-loop feedback policy is denoted as κ^{MPC}.

It is well known that the direct application of the finite receding horizon control obtained from the solution of (3) is not guaranteed to result in a stable closed-loop (see e. g. [Bitmead et al., 1990]). One of the most common strategies to guarantee closed-loop stability and recursive feasibility of an NMPC controller is to design the stage cost L, the terminal cost E and the terminal region so that they fulfill certain assumptions. Different designs are possible (see [Mayne et al., 2000, Grüne and Pannek, 2011, Findeisen and Allgöwer, 2002] and we use in this paper the most common criteria, which is stated in the following assumption.

Assumption 1. (Assumption on the terminal ingredients). The terminal ingredients are designed such that:

i) The stage cost $L(x, u)$ is positive semidefinite with respect to x, positive definite with respect to u and $L(0, 0) = 0$.
ii) The terminal cost $E(x)$ is positive semidefinite
iii) There exist a control policy $\kappa(x)$ inside a control invariant terminal region $\mathbb{X}_f \subseteq \mathbb{X}$ such that:

$$\frac{\partial E}{\partial x} f(x, \kappa(x)) \leq -L(x, \kappa(x)), \forall x \in \mathbb{X}_f$$

The following well known theorem (see e.g. Fontes [2001], Grüne and Pannek [2011], Findeisen et al. [2003]) states the stability of the continous-time NMPC algorithm.

Theorem 1. Under Assumption 1, the system (1) is closed-loop asymptotically stable under the receding control law κ^{MPC}.

Calculating the terminal ingredients that satisfy Assumption 1 is reasonably easy for the linear case (see e. g. [Rawlings and Mayne, 2009]). However, this calculation constitutes one of the main difficulties for the design of NMPC controllers with guaranteed stability and recursive feasibility.

Most of the methods are based on the linearization of the system around an equilibrium point (the origin in this case), obtaining the linearized dynamics $\dot{\tilde{x}} = F\tilde{x} + G\tilde{u}$. Assuming stabilizability of (F, G), a terminal feedback law can be obtained solving the LQR problem using typically a quadratic stage cost of the form $L(x, u) = x^T Qx + u^T Ru$. The obtained feedback law is:

$$u(t) = -Kx(t), \text{with } K = R^{-1}G^T P \tag{4}$$

and P is obtained as:

$$F^T P + PF - K^T RK + Q = 0 \tag{5}$$

Then, as proved in [Chen and Allgöwer, 1998] using the closed loop dynamics $F_K = F + GK$ the following Lyapunov equation has a unique positive-definite and symmetric solution P^*:

$$(F_K + \rho I)^T P^* + P^*(F_K + \rho I) = -Q^*, \tag{6}$$

with $Q^* = Q + K'RK$ and ρ is a non-negative constant that satisfies $\rho < -\lambda_{\max}(F_K)$. The terminal cost is chosen as $E = x^T P^* x$ and as shown by Chen and Allgöwer [1998] there exist a sub-level set of the terminal cost that can be chosen as a terminal set that satisfies the required Assumption 1. The terminal set $\mathbb{X}_f$ is defined as:

$$\mathbb{X}_f(\alpha) = \{x \in \mathbb{X} | x^T P^* x \leq \alpha\}. \tag{7}$$

This methods has several drawbacks. Because of the first order approximation performed for the nonlinear dynamics, for the obtained terminal control and the quadratic approximation of the terminal cost, they can be significantly different from the optimal ones for the nonlinear system. This can lead to a decreased performance, and also to a reduced size of the terminal region. In this paper, we employ a method to achieve an approximation of the optimal feedback and optimal cost of a higher degree, cf. Section 3.