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Simple and efficient moving horizon estimation based on the fast gradient method

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Abstract: By now many results with respect to the fast and efficient implementation of model predictive control exist. However, for moving horizon estimation, only a few results are available. We present a simple solution algorithm tailored to moving horizon estimation of linear, discrete-time systems. In a first step the problem is reformulated such that only the states remain as optimization variables, i.e. process and measurement noise are eliminated from the optimization problem. This reformulation enables the use of the fast gradient method, which has recently received a lot of attention for the solution of model predictive control problems. In contrast to the model predictive control case, the Hessian matrix is time-varying in moving horizon estimation, due to the time-varying nature of the arrival cost. Therefore, we outline a tailored method to compute online the lower and upper eigenvalues of the Hessian matrix required by the here considered fast gradient method. In addition, we discuss stopping criteria and various implementation details. An example illustrates the efficiency of the proposed algorithm.

Keywords: Moving horizon estimation, State estimation, Fast gradient methods.

1. INTRODUCTION

State estimation plays a fundamental role in many applications. It is often elementary for monitoring a system and frequently utilized in combination with a state feedback controller to stabilize a system. Estimation methods based on Kalman filtering, compare Kailath et al. (2000), which are predominantly used in applications, do not allow to easily include constraints on the variables in a structured way. Therefore, Moving Horizon Estimation (MHE) has received increasing interest since it can effectively take constraints on the variables into account. MHE is, similarly as its control "relative" Model Predictive Control (MPC), based on the online solution of an optimization problem over a finite horizon. Therefore, the challenge of an efficient real-time implementation arises, especially for high sampling rates, limited computation power or large scale systems.

In comparison to fast MPC there are only limited results with respect to the efficient implementation of MHE available.

For fast MHE, different approaches have been investigated based on tailored solution approaches for the underlying optimization problem. For nonlinear systems results based on combinations of direct multiple shooting with Gauss-Newton iterations, see (Diehl et al., 2005; Kraus et al., 2006), or sensitivity analysis and nonlinear programming, see e.g. Zavala et al. (2008), exist. Also approximation based methods utilizing for example singular value decompositions, see Jang et al. (2014), or in situ adaptive tabulation, see Abrol and Edgar (2011), have been considered.

Darby and Nikolaou (2007) implemented a look-up table and function evaluation for real-time implementation of MHE for linear systems. Similarly to MPC, however, the number of polytopes generated in the approach tends to grow combinatorially with the number of constraints, which limits the size of the problem that can be handled. Haverbeke et al. (2009) developed a primal barrier interior-point method algorithm for linear system exploiting the system structure.

For moving horizon estimation of constrained, linear, discrete-time systems we propose a simple, tailored algorithm based on Nesterov's fast gradient method (Nesterov, 1983, 2004). Fast gradient methods have recently received considerable attention for the solution of optimization problems arising in model predictive control, see e.g. Faulwasser et al. (2014); Jerez et al. (2014); Kögel and Findeisen (2011); Patrinos and Bemporad (2014); Richter et al. (2012, 2010); Zometa et al. (2012).

To enable the efficient solution of the MHE problem using the fast gradient method we propose to eliminate the optimization variables related to the noise from the optimization problem. This results in a sparse formulation with only the states as optimization variables. In contrast to MPC, in MHE the Hessian matrix and thus also its eigenvalues are time-varying due to the arrival cost. Since the fast gradient method requires the largest and smallest eigenvalues of the Hessian matrix (or tight bounds on them), we discuss how to efficiently compute these based on the so-called inverse iteration, (Golub and Van Loan, 2012). Additionally, we review stopping criteria and discuss the implementation of the proposed algorithm. The applicability and performance of the presented method is illustrated by an example.

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The remainder of this work is structured as follows. Section 2 presents the problem formulation. In Section 3 we illustrate how to formulate the optimization problem by using only the states as optimization variables. Section 4 gives a detailed description of the proposed solution method. In Section 5 we illustrate the results. Finally, we provide a summary and outline future working directions.

2. PROBLEM FORMULATION

This section presents the class of considered systems and outlines the moving horizon estimation procedure.

2.1 Considered problem class

Time-invariant, discrete-time, linear systems of the form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \\ y_k &= Cx_k + v_k. \end{aligned} \tag{1}$$

are considered, were $k \geq 0$ denotes the time, $x_k \in \mathbb{R}^n$ the state, $u_k \in \mathbb{R}^m$ the known input, $w_k \in \mathbb{R}^n$ the unknown process noise, $y_k \in \mathbb{R}^p$ the observed output and $v_k \in \mathbb{R}^p$ the unknown measurement noise. The matrices have appropriate dimensions and (A,C) is assumed to be detectable.

We assume that the state x_k and the measurement noise v_k are each constrained to polytopic, compact, convex sets: $x_k \in \mathbb{X}$ and $v_k \in \mathbb{V}$. Often, process noise w_k and measurement noise v_k are assumed to be (approximately) zero mean, Gaussian white noise with covariances given by Q and R, respectively, compare Rawlings and Mayne (2009). For the initial state x_0, an estimate $\tilde{x}_0$ with covariance Π_0 is available at $k = 0$. Note, while this provides for MHE somehow a relation to the stochastic optimal Kalman Filter, a pure deterministic setting is also possible. We assume that Q, R and Π_0 are positive definite.

Remark 1. (Neglection of bounds on the process noise w_k) *Considering bounds only on the state and measurements is motivated by many practical applications: constraints on the state x can represent physical boundaries (*e.g. *concentrations, temperature, pressures, liquid level) and constraints on the measurements noise v_k can be obtained from the specification of the sensors (*e.g. *maximum error). In contrast, it is often more difficult to obtain bounds on the process noise w_k, except in a few special cases such as e.g. in (Rausch et al., 2014).*

2.2 Moving horizon estimation

To estimate the state x_k, $k > 0$ of system (1) based on the information available before the time instant k, one can use moving horizon estimation, see (Rao et al., 2001; Rawlings and Mayne, 2009) and the references therein for more details. The optimization based nature of MHE allows to take the bounds on x_k and v_k into account; in contrast to classical estimation techniques such as Kalman filtering, see Kailath et al. (2000); Rao et al. (2001); Rawlings and Mayne (2009).

The general idea in MHE to estimate x_k, c.f. Rawlings and Mayne (2009), is to solve at each time instant

$$\begin{aligned} &\min_{\hat{\mathbf{v}},\hat{\mathbf{w}},\hat{\mathbf{x}}} J(\hat{\mathbf{v}},\hat{\mathbf{w}},\hat{\mathbf{x}},\tilde{x}_s,\Pi_s) \\ &\text{s.t. } \hat{x}_{j+1} = A\hat{x}_j + Bu_j + \hat{w}_j \\ &\qquad y_j = C\hat{x}_j + \hat{v}_j \\ &\qquad \hat{x}_i \in \mathbb{X},\ \hat{v}_j \in \mathbb{V}, \end{aligned} \tag{2}$$

where $s = \max(0, k-N)$ denotes the start of the estimation window, N is the (maximum) estimation window size, also called estimation horizon, $i = s,\ldots,k$, $j = s,\ldots,k-1$ and $\hat{\mathbf{x}} = \{\hat{x}_i\}$, $\hat{\mathbf{v}} = \{\hat{v}_j\}$, $\hat{\mathbf{w}} = \{\hat{w}_j\}$, $\mathbf{u} = \{u_j\}$ and $\mathbf{y} = \{y_j\}$. $\tilde{x}_s$, $s > 0$, is the prior estimate of x_s using the information available prior to $k = s$. The cost function J is given by

$$\begin{aligned} J(\hat{\mathbf{v}},\hat{\mathbf{w}},\hat{x}_s,\tilde{x}_s,\Pi_s) =& \frac{1}{2}\sum_{i=s}^{k-1} \|\hat{w}_i\|^2_{Q^{-1}} + \frac{1}{2}\sum_{i=s}^{k-1} \|\hat{v}_i\|^2_{R^{-1}} \\ &+ \frac{1}{2}\|\tilde{x}_s - \hat{x}_s\|^2_{\Pi_s^{-1}}, \end{aligned} \tag{3}$$

where the choice of the so-called arrival cost matrix Π_s, which weights the influence of the estimate $\tilde{x}_s$ for $i > 0$, is discussed below. As an estimate for x_k at each time, the optimal value of $\hat{x}_k$ from (2) is used: $\tilde{x}_k = \hat{x}_k^\star$ [1].

The choice of the estimation windows size N is a trade-off between the maximum size of the optimization problem (2) and the estimation performance: using a smaller N reduces the computational demand, but can lead to deteriorated estimates.

Note that the optimization problem (2) is a convex, quadratic program, which needs to be solved at every time-instance, so an efficient solution is of key interest. Therefore, we present in the following sections a tailored solution approach.

Remark 2. (Arrival cost matrix Π_i, choice of $\tilde{x}_i$) *The choice of the arrival cost matrix Π_i and the choice of $\tilde{x}_i$ are crucial for a good estimation performance. Incorrect choices can lead to inferior estimation and even an unstable estimation error dynamics, see (Rao et al., 2001; Rawlings and Mayne, 2009) for more details. We consider here only the simple approach using the prior state estimate $\tilde{x}_i$ as estimate of x_i computed at $k = i-1$ and to update Π_i using a Kalman filtering update*

$$\Pi_{k+1} = A\Psi_k A^T + Q, \tag{4a}$$

$$\Psi_k = \Pi_k - \Pi_k C^T (C\Pi_k C^T + R)^{-1} C\Pi_k. \tag{4b}$$

Note that Π_k will converge to a unique fix point Π_∞, where the matrix Π_∞ as well as Π_i are positive definite, because (A,C) is detectable, $(A, W^{\frac{1}{2}})$ is stabilizable and V, Π_0 are positive definite, compare Kailath et al. (2000).

Remark 3. (Feasibility of MHE problem (2)) *Since w_k is not constrained, the arising optimization problem is always feasible under the assumptions made, i.e. that the measurements are consistent with $\mathbb{V}$ and $\mathbb{X}$: for every y_k there exists x_k and v_k such that $y_k = Cx_k + v_k$, $x_k \in \mathbb{X}$ and $v_k \in \mathbb{V}$. There is always a w_k such that* (1) *is satisfied.*

3. REFORMULATION OF THE OPTIMIZATION PROBLEM

We first formulate the MHE optimization problem (2) such that the optimization is performed only over the state trajectory $\hat{\mathbf{x}}$, i.e. eliminating the noise sequences $\hat{\mathbf{w}}$ and $\hat{\mathbf{v}}$. This idea is inspired by similar ideas exploited in MPC, c.f. Mancuso and Kerrigan (2011), where the inputs are eliminated from the control variables to increase the speed of interior point methods.

First one can straightforwardly eliminate the measurement noise $\hat{\mathbf{v}}$ as optimization variable, see Haverbeke et al. (2009), by replacing $\|\hat{v}_i\|^2_{V^{-1}}$, $\hat{v}_i \in \mathbb{V}$ by $\|y_i - C\hat{x}_i\|^2_{V^{-1}}$ and $y_i - C\hat{x}_i \in \mathbb{V}$, respectively. The resulting optimization problem possess as

[1] Note that we consider here an estimate $\hat{x}_k$ using only information available prior to the time instance k: the so-called predicted or a priori estimate, compare Kailath et al. (2000). The proposed approach can be extended such that also y_k is used, i.e. to obtain the a posteriori (also called filtered) state estimate.