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CONTROLLABILITY PROPERTIES IN SAFE REGIONS ⋆

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Abstract: For nonlinear control systems in discrete time, the global controllability structure within a safe region of the state space is analyzed. The main results characterize those safe regions, where every point can be steered into a relatively invariant subset of complete approximate controllability. Furthermore, for parameter dependent systems, loss of invariance is analyzed.

1. INTRODUCTION

The purpose of this paper is to analyze controllability properties of nonlinear systems under the additional requirement that a prescribed safe region W in the state space M is not left. The safe region corresponds to the requirement that the system should satisfy certain constraints in order to ensure integrity of the system. Thus, if a trajectory leaves the safe region W, then the system stops. Another interpretatioen is that the complement $H := M \setminus W$ of the world W is a hole in the state space, through which the system may disappear. In the theory of (uncontrolled) dynamical systems, one speaks of "open dynamical systems", or systems with holes in the state space, and there is a considerable body of literature on them, cf. Demers and Young [2006] for a survey. In the present paper we will analyze a class of open control systems in discrete time. A central notion here are control sets, i.e., maximal subsets of complete approximate controllability. For systems in discrete time, Albertini and Sontag were the first to study control sets, cf. Albertini and Sontag [1993]. Control sets and their relations to flows and semiflows have also been analyzed by San Martin and coworkers in the context of semigroups in Lie groups. They elucidate relations between the structure of semisimple Lie groups, semigroup actions, and control sets; cf. e.g. Patrao and San Martin [2007]. Parameter dependence of control sets has been analyzed in Gayer [2004], Graf [2011], and Colonius and Kliemann [2000].

The invariant subsets of complete controllability are called the invariant control sets. If, under small perturbations, invariance is lost, one may expect that the perturbed system still shows similar, although transient behavior. Here the invariant control sets turn into control sets which are no more invariant and one can show that they lose their invariance only if they change discontinuously in the Hausdorff metric; cf. [Gayer, 2004, Corollary 24]. In the present contribution, we show that control sets relative to a safe region, called here W-control sets, are generated. If the safe region is small enough, the generated W-control set is invariant relative to W. Thus the invariant control sets turn into relatively invariant control sets.

The original motivation for the analysis of invariant control sets (in continuous time) is due to the fact, that they often determine the support of invariant measures of associated stochastic systems; cf. Arnold and Kliemann [1987]. First results which indicate that analogously relatively invariant control sets often determine the supports of conditionally invariant measures for associated stochastic system are given in Colonius [2012]. This provides a major motivation for the present paper. It is worth to mention that the analysis and computation of safe regions in control systems also originates from rather different application areas. For example, Tomlin et al. [1998] discuss safe regions motivated by aircraft traffic control problems.

The contents of this paper is as follows: In Section 2, relative control sets and their invariance properties are characterized. Section 3 discusses loss of invariance under parameter changes. Furthermore, a simple example is discussed which illustrates the results.

2. CONTROL SETS AND RELATIVE INVARIANCE

In this section, basic definitions and properties of control systems in discrete time are collected and results on relative invariance for deterministic control systems are proved. Some results in the continuous time case have been given in Colonius and Kliemann [2000], for the discrete time case considered here we rely on Patrao and San Martin [2007] and Colonius et al. [2010]; cf. also Wirth [1998].

Suppose a discrete time control system on a state space M is given which has the form

$$x_{k+1} = f(x_k, u_k),\ u_k \in \Omega, k \in \mathbb{Z}, \tag{1}$$

where M is a subset of $\mathbb{R}^d$ (or a manifold), $\Omega \subset \mathbb{R}^d$ is compact and connected with $\overline{\mathrm{int}\Omega} = \Omega$ and $f : M \times \Omega \to M$ is a continuous map. Throughout we also assume that $f_v := f(\cdot, v)$ is a diffeomorphism on a neighborhood of M for every $v \in \Omega$. Suppose that an open, relatively compact subset $W \subset M$ is fixed such that $f(W \times \Omega) \cap W \neq \emptyset$ and

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$f(W\times\Omega)\not\subset W$. We may think of the prescribed region W as the world in which the system lives.

Sometimes, the following notation will be useful: Let $f_W := f_{|W\times\Omega} : W\times\Omega \to M$, and consider, with a slight abuse of notation, the following "open" control system

$$x_{k+1} = f_W(x_k, u_k),\ u_k \in \Omega, \tag{2}$$

Note that (2) only makes sense, if $x_k \in W$. Thus this system may enter $M\setminus W$, but it cannot leave $M\setminus W$.

For $x\in M$ and a control function $u : \mathbb{N}\cup\{0\}\to\Omega$ we abbreviate $f^0(x,u) := x$ and

$$f^n(x,u) := f_{u_n}\circ f_{u_{n-1}}\circ ...\circ f_{u_0}(x), n\in\mathbb{N}.$$

Analogously, the restricted maps $f^n_W(x,u)$ are defined. Control system (1) is forward accessible in W if for every $x\in W$ and every $n\in\mathbb{N}$ the reachable sets or positive orbits relative to W

$$\mathcal{O}^{+,n}_W(x) = \{f^n_W(x,u)\mid u:\mathbb{N}\cup\{0\}\to\Omega\}$$

have nonvoid interiors. Obviously, this holds iff all reachable sets $\mathcal{O}^{+,1}_W(x), x\in W$, at time $n=1$ have nonvoid interiors. Furthermore, forward accessibility implies that for every $x\in W$ there is a control u with $f^n(x,u)\in W$ for all $n\in\mathbb{N}$. We also define the negative orbits relative to W

$$\mathcal{O}^{-,n}_W(x) := \{y\in W\mid\ x = f^n_W(y,u), u:\mathbb{N}\cup\{0\}\to\Omega\}$$

Throughout the rest of the paper, we restrict attention to forward accessible control systems of the form (2) with the additional property that the negative relative orbits $\mathcal{O}^{-,1}_W(x), x\in W$, are either empty or have nonvoid interior. For brevity, we just call these systems accessible.in the open, relatively compact world W in a state space M. We also write

$$\mathcal{O}^+_W(x) := \bigcup_{n\in\mathbb{N}}\mathcal{O}^{+,n}_W(x) \text{ and } \mathcal{O}^-_W(x) := \bigcup_{n\in\mathbb{N}}\mathcal{O}^{-,n}_W(x).$$

Restricting attention to the world W, we obtain the following notions.

A subset $D_W\subset W$ with nonvoid interior is called a W-control set (or relative control set with respect to W) if for all $x,y\in D_W$ one has $y\in\overline{\mathcal{O}^+_W(x)}$ and D_W is maximal with this property i.e., if $D'_W\supset D_W$ is a set such that $y\in\overline{\mathcal{O}^+_W(x)}$ for all $x,y\in D'_W$, then $D_W = D'_W$. A W-control set is called relatively invariant, if $x\in D_W$ and $f^k(x,u)\notin D_W$ for some control u and some $k\in\mathbb{N}$, implies $f^k(x,u)\notin W$.

For the sake of brevity, we call relatively invariant W-control sets just relatively invariant control sets, if it is clear from the context, which world W is considered. If $W=M$, we omit the index W and just speak of control sets and invariant control sets. By accessibility, a subset $D\subset M$ is an invariant control set iff $\overline{\mathcal{O}^+(x)} = D$ for all $x\in D$. Furthermore, a control set (with $W=M$) as defined above is also a control set in the sense of [Patrao and San Martin, 2007, Section 4.2] (where a much more general situation is considered) and hence all properties derived in that paper hold for control sets as defined above. In particular,

$$D = \overline{\mathcal{O}^+(x)}\cap\mathrm{int}\mathcal{O}^-(x)$$

for every x in the core (or transitivity set as it is called in [Patrao and San Martin, 2007, Section 4.2]) defined by $\mathrm{core}D := \{y\in D\,|\,\text{there is } z\in D \text{ with } z\in\mathrm{int}\mathcal{O}^+(y)$ and $y\in\mathrm{int}\mathcal{O}^+(z)\}$; the set core$D$ is open and it is dense in D. Since Ω is connected, a control set is invariant iff it is closed; see [Colonius et al., 2010, Lemma 3]. Relative control sets are, in general, properly contained in control sets, since they need not be maximal with respect to the whole state space. Nevertheless, they enjoy many properties which are analogous to those of control sets.

We define the core of a relatively invariant control set D_W by $\mathrm{core}D_W := \{y\in D_W\,|\,\text{there is } z\in D_W \text{ with } z\in\mathrm{int}\mathcal{O}^+_W(y) \text{ and } y\in\mathrm{int}\mathcal{O}^+_W(z)\}$.

Theorem 1. (i) Relative control sets are pairwise disjoint. (ii) For every relative control set D_W the core coreD_W is an open set and it is dense in D_W. (iii) A relative control set D_W is relatively invariant iff it is closed relative to W. (iv) Let D_W be a relatively invariant control set. Then D_W is an invariant control set iff $\partial D_W\cap\partial W = \varnothing$.

Proof. Assertions (i) to (iii) follow by minor modifications of the proofs in [Patrao and San Martin, 2007, Section 4.2] and [Colonius and Kliemann, 2000, Section 3.3]. If the condition $\partial D_W\cap\partial W=\varnothing$ in (iv) holds, assertion (iii) implies that D_W is a closed control set in M and hence an invariant control set for system (1). The converse follows, since an invariant control set in M is closed.

The main result on existence of relatively invariant control sets is the following.

Theorem 2. Consider a control system of the form (2) which is accessible in an open, relatively compact world W in a state space M. Consider $x\in W$ and assume that there exists a closed set $Q\subset W$ such that for all $y\in\mathcal{O}^+_W(x)$ one has $\overline{\mathcal{O}^+_W(y)}\cap Q\neq\emptyset$. Then there exists a relatively invariant W-control set $D_W\subset\overline{\mathcal{O}^+_W(x)}$. Furthermore, the following assertions are equivalent: (i) There is a closed set $Q\subset W$ such that $\overline{\mathcal{O}^+_W(x)}\cap Q\neq\emptyset$ for all $x\in W$. (ii) For every $x\in W$ there is a relatively invariant control set D with $D\subset\overline{\mathcal{O}^+_W(x)}$. If (i) holds, there are only finitely many relatively invariant control sets.

Proof. For $y\in\overline{\mathcal{O}^+_W(x)}$ let $Q(y) := \overline{\mathcal{O}^+_W(y)}\cap Q$. Consider the family $\mathcal{F}$ of nonvoid and compact subsets in W given by $\mathcal{F} = \{Q(y),\ y\in Q(x)\}$. Then $\mathcal{F}$ is ordered via

$$Q(y)\prec Q(z) \text{ if } z\in\overline{\mathcal{O}^+_W(y)}.$$

Every linearly ordered set $\{Q(y_i),\ i\in I\}$ has an upper bound

$$Q(y) = \bigcap_{i\in I}Q(y_i) \text{ for some } y\in\bigcap_{i\in I}Q(y_i),$$

because the intersection of decreasing compact subsets of the compact set Q is nonempty. Thus Zorn's lemma implies that the family $\mathcal{F}$ has a maximal element $Q(y)$. Now we claim that the set

$$D_W := \overline{\mathcal{O}^+_W(y)}^W$$

is a W-invariant control set; here the closure is taken relative to W. In fact: Note first that $y\in Q\subset W$, hence $y\in D_W$ and by accessibility

$$\emptyset\neq\mathrm{int}\mathcal{O}^+_{W,\leq t}(y)\subset D_W.$$

Thus int$D_W\neq\emptyset$. Furthermore, every $z\in D_W$ is approximately reachable from y within W, i.e., $D_W\subset\overline{\mathcal{O}^+_W(y)}^W$.