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Opinion consensus under external influences[☆]

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ABSTRACT

As a means to regulate the continuous-time bounded confidence opinion dynamics, an exo-system to the original Hegselmann–Krause model is added. Some analysis is made about the properties of the combined system. Two theorems are provided in this article in terms of sufficient conditions of the exo-system that can guarantee opinion consensus for any initial conditions. Two more corollaries are given to describe the resulting synchronized opinions.

1. Introduction

Self-organized group behavior can be observed in animal flocks as well as human society. By simple communication, a whole group of birds can react to danger rapidly, or can migrate in a certain formation. A herd of predator can hunt with special tactics by following certain self-organized rules. As higher intelligent and relatively more independent human beings, we think and behave based on our own will while influenced by many other human individuals and by the society. Sociologists, psychologists, and even engineers and mathematicians want to model and study this interaction among human societies nowadays. While psychologists focus more on how people handle and react from social input, mathematicians analyze relatively simple models and the resulting group/global behaviors.

Among the different models about human opinion dynamics in the literature, there is a type of model called bounded confidence model who allows opinion influence to happen only when the two opinions are close enough. This type of modes also has the name “Hegselmann-Krause models” from its initiators Rainer Hegselmann and Ulrich Krause [1]. These discrete-time, deterministic models force the opinion of an individual to reach the opinion average among its close-by neighbors at every time step. The models lead to a well known clustering phenomenon that the opinions will converge to one or a few certain constant values. For a given model, the number and position of those values are completely determined by the starting opinion distributions.

Despite the simple expression of the Hegselmann–Krause models, the analysis of the opinion evolution is complicated. Besides the original article by Hegselmann and Krause, there are papers such as [2–4] that study the convergence, stability and steady state of the opinions under the Hegselmann–Krause model. By applying the theory of differential equations, the model is also extended to continuous-time opinions; [5,6] provide a detailed analysis on the existence and uniqueness of the solution to the continuous-time model, which is non-trivial due to the discontinuous right-hand side.

A frequently asked question is: for what initial opinion distribution the system will have only one cluster when time evolves. Reaching only one cluster is also called reaching consensus. The sufficient and necessary condition for reaching consensus has not been given in the literature. There are a few results for sufficient conditions such as those in [7]. Other researchers test some modified version of the Hegselmann–Krause model for different purposes. Those models can be found in [7,8]. In [9], a bounded confidence model with antagonistic interactions was discussed based on the analysis of the original Altafini’s model in [10–12].

Instead of the standard homogeneous Hegselmann–Krause models, heterogeneous models that consider agents as different individuals are studied in the literature [13,14], which can improve the chance of reaching consensus for random initial conditions. An example of the heterogeneous models is by introducing “stubborn agents” that are unwillingly to change their opinions [15]. The extreme case of those stubborn agents can be considered as the state of a special type of exo-system that has zero dynamics. In this paper, we combine the Hegselmann–Krause model with a general exo-system. If the exo-system satisfies certain sufficient conditions, the consensus behavior can be guaranteed for any random initial opinion distribution.

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The Hegselmann–Krause model with exo-systems is introduced in Section 2 with some properties of the model. Two theorems are provided that guarantee consensus in Section 3 together with two corollaries. In Section 4, numerical experiments are carried out to test and illustrate the results of the theorems and corollaries. A short summary is included in Section 5 as well as a forecast of possible future study.

2. Problem formulation

We study a system of N agents with their time-dependent opinions denoted as $x_i(t) \in \mathbb{R}$ for $i = 1, 2, \dots, N$. Agent i is influenced by agent j only if the opinions of both are close enough. The dynamics of the opinions can be modeled as the following system:

$$\dot{x}_i = \sum_{j:|x_j - x_i|<d} (x_j - x_i), \tag{1}$$

where the distance $d > 0$ is called the confidence range. The system (1) is also referred to as continuous-time Hegselmann–Krause (H–K) model initially introduced in [5] based on the original discrete-time "bounded confidence" models. There are both theoretical analysis and numerical simulations about this model in the literature, showing the cluster behavior of the opinions. Asymptotically, each opinion will converge to one of the several certain values called opinion clusters. The distance between any pair of opinion clusters is proven to be larger than d in [5].

If the opinions converge to a single cluster, we say the opinions reach consensus. There are a few results about consensus behavior for both system (1) and modified versions of (1) with given initial conditions. For an arbitrary random initial opinion distribution, consensus is however not guaranteed even if the neighbor graph is initially connected. In this paper we introduce certain exo-systems in addition to the H–K model so that opinion consensus can be reached for a broader range of initial conditions. The new system is modeled as follows:

$$\begin{aligned} \dot{x}_i &= \sum_{j:|x_j - x_i|<d} (x_j - x_i) + \sum_{k:|y_k - x_i|<d} (y_k - x_i) \\ \dot{y}_k &= g_k(t, y), \end{aligned} \tag{2}$$

for $i = 1, 2, \dots, N$ and $k = N+1, N+2, \dots, N+M$, where y_k's are the state variables for the exo-system and g_k's are bounded continuous functions that we design later. M is dimension of the exo-system. For simplifying later use, $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$ will denote the stack vectors of x_i's and y_k's, respectively. In the context, we also call the state of the original H–K model x_i's normal opinions in order to separate them from the external opinion y_k's.

Because of the discontinuity of the right-hand side of the continuous-time H–K model, the classic analysis of the existence and uniqueness of the solution does not apply. There are many discussions in the literature about different types of solutions to this differential equation. In [5], the authors introduced the concept of "proper solutions" as a subset of Carathéodory solutions, which guarantee existence and uniqueness for "proper initial conditions" that are almost sure in measure. In [6], the more general Krasovskii solutions are discussed for H–K models, which exist for any initial condition without a guarantee of uniqueness. There is a detailed discussion about the difference between these two types of solutions at the end of [6]. In our case, as long as the g function in the exo-system is locally bounded and well defined in an open neighborhood around the initial time, the existence of Krasovskii solution will be guaranteed (see [16]). Since g is given for design purposes, we assume that in this paper $g_k(t, y)$ is continuous and is defined for all $t \in \mathbb{R}$, and thus we consider that $(x(t), y(t))$ is a Krasovskii solution to (2) for the rest of the paper. Similar to the approach introduced in [6], we use the following way to describe a solution.

Given a Krasovskii Solution $(x(t), y(t))$, define the joint graph $G(x, y) = (V_x \cup V_y, E(x, y))$, where $V_x = \{1, 2, \dots, N\}$ and $V_y = \{N+1, N+2, N+M\}$ are the sets of vertices, and

$$\begin{aligned} E(x, y) = &\{(i, j) : i, j \in V_x, |x_i - x_j| < d\} \\ &\cup \{(i, k) : i \in V_x, k \in V_y, |x_i - y_k| < d\} \end{aligned}$$

is the set of edges.[1] Slightly different from the standard graph theory definition, in this paper, we call two opinions *connected* if and only if their distance is less than d, i.e., i, j are connected if and only if $(i, j) \in E(x, y)$ for $i, j \in V_x \cup V_y$. We also need to define the concept of boundary of the edge set, which is

$$\begin{aligned} \partial E(x, y) = &\{(i, j) : i, j \in V_x, |x_i - x_j| = d\} \\ &\cup \{(i, k) : i \in V_x, k \in V_y, |x_i - y_k| = d\}. \end{aligned}$$

$\partial E(x, y)$ includes those pairs (i, j) that have the exact distance d for the corresponding opinions. The pair should not be in the set of edges but with any arbitrary small perturbation in the negative direction, it will form an edge between them. These pairs are the locations of the potential discontinuities in the system, where the Krasovskii solution allows small perturbation around them. Note that the dynamics of the exo-system is always continuous as we assumed, we only focus on the normal opinions $x_i(t)$.

By definition, for any given Krasovskii solution $(x(t), y(t))$ and for almost any time t, it holds that $\frac{d}{dt}(x(t), y(t))$ belongs to the closed convex hull of intersecting the right-hand-side of (2) with arbitrary small perturbation of the states. Since we have already defined the boundary of the edge set, we can always find a set of normalized weights $\alpha_H^{(x,y)}$, depending on both the solution (x, y) and the time t, such that

$$\frac{d}{dt} x_i(t) = \sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} \times \left(\sum_{j \in \mathcal{N}_i^H} (x_j(t) - x_i(t)) + \sum_{k \in \tilde{\mathcal{N}}_i^H} (y_k(t) - x_i(t)) \right), \tag{3}$$

where $\alpha_H^{(x,y)} > 0$, $\sum_{H \subset \partial E(x,y)} \alpha_H^{(x,y)} = 1$, and $\mathcal{N}_i^H$ and $\tilde{\mathcal{N}}_i^H$ are defined as follows, respectively:

$$\mathcal{N}_i^H = \{j \in V_x : (i, j) \in E(x, y) \cup H\};$$

$$\tilde{\mathcal{N}}_i^H = \{k \in V_y : (i, k) \in E(x, y) \cup H\}.$$

H denotes the possible set of edges that can be added to the graph, where the length of each of those edges is exactly d. Namely, the two opinions that have difference d can sometimes be considered as directly connected. We also define $\mathcal{N}_{ij}^H = \mathcal{N}_i^H \cap \mathcal{N}_j^H$, and $\tilde{\mathcal{N}}_{ij}^H = \tilde{\mathcal{N}}_i^H \cap \tilde{\mathcal{N}}_j^H$ for the simplification of later uses.

We start our analysis of the model by introducing the following order preservation property (Proposition 2.1) first.

Remark. It is intuitive to consider the order preservation property in the following way: when $x_i(t) = x_j(t)$ for some t, they should retain the same opinion for $t' > t$ since the derivative is only state related. However, this property only holds for "proper solutions" as shown in [5]. For general Krasovskii solutions, the equality for x_i and x_j may be lost at discontinuities even after time t [6]. Therefore, the following order preservation property only covers the cases of strict inequality.

[1] The graph is defined at time t and is time dependent. To simply the notation, we do not add t explicitly to the definitions. The edge definition here makes the graph a directed graph. However, the interaction between normal opinions is still symmetric.