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On structural behavioural controllability of linear discrete time systems with delays

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ABSTRACT

In this paper we study the controllability of interconnected networks that are described by means of structured linear systems with state-like and control variables. We assume that the systems operate in discrete time with the set of integers as the time axis. Further, we assume that the state-like variables for their evolution only depend on recent values of their neighbours with, however, unknown weight factors. These recent values may be one step back in time, but also more steps. This yields a description of the systems by means of matrices containing fixed zeros and free parameters, together with a time lag structure. Knowing the dependency and lag structure, we represent (the structure of the) systems by means of weighted directed graphs and study questions concerning their structural controllability, where the latter has to be defined in an appropriate way, i.e., in behavioural sense. We provide a necessary and sufficient characterization of structural controllability of our systems using a graph representation. The obtained characterization makes use of well-known and efficient algorithms from graph theory. We prove that in this context finding the minimal number of driver (controller) nodes is an NP-hard problem. The concepts and results of the paper are illustrated on academic examples and on a gene regulatory network.

1. Introduction

We consider complex networks in *discrete* time that are generalizations of the classical linear discrete time systems $x(k+1) = Ax(k) + Bu(k)$.

We study here models that are discrete in time because in many applications the evolution of processes is described by discrete steps in time, rather than by a continuously changing time. For instance, think of evolution models from economy, social sciences, politics, biology, see [1] and [2]. Discrete time models are also obtained by sampling continuous time processes at discrete times, and controlling them by means of piecewise constant controls.

The networks underlying our systems will in general have a non-linear description. However, here we assume the equations of our models to be linear. The reason for this is that the original network equations in many cases can be approximated by linear ones as long as the solutions stay close to the point of linearization. In such a linearization, the original dependency structure often remains conserved, i.e., the mutual dependency structure of the variables often does not change by the linearization, only the non-linear equations are replaced by linear ones. The same applies to the time lag structure in the non-linear equations, implying that also the time lag structure is preserved in the linearization process. However, the intensity by which the variables influence each other may be unknown after doing the linearization due to the fact that the point of linearization may not be known precisely, and/or the fact that the actual non-linear description of the intensities in the original equations is not known exactly.

Furthermore, in addition to the linearity, we assume that the next value of each state-like variable depends only on the recent value of a number of neighbouring state-like and control variables. We assume that each of these variables occurs at most once in the equation of the next value of a state-like variable, with a given lag and a non-zero, but otherwise unknown coefficient. For continuous-time systems, Morse in [3] introduced a ring model to deal with systems with (commensurate) delays and studied the controllability for this type of system.

Like often in behavioural theory [4], in this paper we assume that the systems have started a long time ago. In particular, we do not allow for initial conditions, i.e., values of state-like variables, that are not the result of some previous behaviour. Initial conditions should make sense and should be realistically achievable from the past. Therefore, we assume that the time axis is the set of

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integers, rather than the set of natural numbers, i.e., non-negative integers. In other words, we do not consider our systems to have a beginning in time, but to be a part of an evolution that started at minus infinity.

It turns out that the classical notion of controllability may not apply and that the notion of controllability in the behavioural sense of Willems is more appropriate. The goal in this paper is to find methods to establish the controllability of structural versions of the system under consideration, derived from their weighted graphs. Indeed, the main result of the paper are simple necessary and sufficient graph-theoretic conditions for structural controllability in behavioural setting for the type of systems mentioned above.

Our result does however not say anything about systems defined on only non-negative integer times, or systems where in the right hand-side of the evolution equations components may occur with multiple time delays. For more on the latter type of systems, their controllability, related approaches and results, see [5] and [6]. An approach to get structural controllability results for the latter type of systems, could be one using a state space representation in which the state-like variables are seen as the outputs. The problem then becomes an output controllability question. The (difficult) output controllability problem for structured systems was studied, but left as an open problem, in [7].

As indicated, our main result are simple necessary and sufficient conditions for structural controllability in behavioural setting for the type of studied in this paper. When not present, from the proof of our result, also a selection of input nodes can be extracted to guarantee the controllability. However, this selection is not necessarily as small as possible. The so-called *minimal controllability problem* (MCP) received a great deal of attention in the last decade. Roughly speaking, starting with an autonomous system (i.e., without inputs), one looks for adding a minimum number of inputs to make the system controllable. For controllability in the usual sense, the MCP was studied in some classical papers [8–11] for different variants of this problem. Here we prove that the MCP in behavioural sense, for structured systems with delays, is NP-hard. However, for an important class of systems, often met in practice, we show that the controllability in behavioural sense can be checked in polynomial time.

The outline of this paper is as follows. In Section 2 we present the type of linear discrete time system that we study in this paper. Also we introduce the type of controllability that best fits the situation in this paper and recall a controllability condition together with its relationship with some classical notions of controllability. In Section 3 we introduce the graph by which the structure of the systems can be represented. Also we present some notions of graph theory by which the main result of this paper can be presented. We state the main result in Section 4. This section also contains a proof of the result. Section 5 deals with computational aspects in relation to the main result. The minimal controllability problem (MCP) for such systems is addressed in Section 6. The main result is illustrated in Section 7 by means of some examples. Finally, in Section 8 we conclude the paper with some remarks and topics for future research. In the reference list background material can be found.

2. Problem statement

We consider in this paper the class of linear discrete time systems with dynamics described by

$$x_i(k+1) = \sum_{j\in\mathcal{A}_i} a_{ij}x_j(k-p_{ij}) + \sum_{j\in\mathcal{B}_i} b_{ij}u_j(k-q_{ij}), \quad i = 1,\dots,n, \tag{1}$$

where $k \in \mathbb{Z}$. In (1), $\mathcal{A}_i$ is the index set of non-zero coefficients in the ith equation, and likewise for $\mathcal{B}_i$. In the sequel, the coefficients a_{ij} and b_{ij} are assumed to be constant. Later on, these coefficients are even assumed to have an unknown value. Further in (1), $x_j(k)$ denotes the value at time $k \in \mathbb{Z}$ of the jth component of the state-like vector $x(k) \in \mathbb{R}^n$. Similarly, $u_j(k)$ denotes the value at time $k \in \mathbb{Z}$ of the jth component of the control vector $u(k) \in \mathbb{R}^m$. Finally in (1), p_{ij} and q_{ij} are non-negative integers indicating the lag of x_j and u_j, respectively, in the ith equation.

Denoting the backward time shift by σ (i.e., $\sigma x(k) = x(k+1)$), system (1) can be written as

$$\sigma x(k) = \mathbb{A}(\sigma^{-1})x(k) + \mathbb{B}(\sigma^{-1})u(k), \tag{2}$$

with

$$\left(\mathbb{A}(z^{-1})\right)_{ij} = \begin{cases} a_{ij}z^{-p_{ij}} & 1 \leqslant i \leqslant n, j \in \mathcal{A}_i, \\ 0 & \text{else,} \end{cases}$$

and

$$\left(\mathbb{B}(z^{-1})\right)_{ij} = \begin{cases} b_{ij}z^{-q_{ij}} & 1 \leqslant i \leqslant n, j \in \mathcal{B}_i, \\ 0 & \text{else.} \end{cases}$$

Clearly, $\mathbb{A}$ and $\mathbb{B}$ are $n \times n$ and $n \times m$ polynomial matrices, respectively, in the indeterminate z^{-1}. The pair $(x(k), u(k))$ is said to be a solution of (2) if the vectors $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ satisfy (2) for all $k \in \mathbb{Z}$, i.e., the system is supposed to be defined for all integer times, also negative ones.

The above class is more general than the class of classic linear discrete time systems

$$\sigma x(k) = Ax(k) + Bu(k), \tag{3}$$

with $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $\sigma x(k) = x(k+1)$, and A and B matrices of appropriate dimensions.

When the variable $x(k)$ is a true state, i.e., satisfies the axioms of state, like in (3), the classical notion of *controllability* means that for any $x_0, x_1 \in \mathbb{R}^n$ there exists a time τ and a control sequence $u(k)$, $0 \leqslant k < \tau$ such that the state is steered from $x(0) = x_0$ to $x(\tau) = x_1$. When the final state x_1 is replaced by zero, we obtain the notion of *zero controllability*, i.e., for any $x_0 \in \mathbb{R}^n$ there exists a time τ and a control sequence $u(k)$, $0 \leqslant k < \tau$ such that the state is steered from $x(0) = x_0$ to $x(\tau) = 0$.

It is well-known that the above two notions of classical controllability are not equivalent when dealing with discrete time systems. Indeed, there holds, see [12] or [13]:

Theorem 1. *System (3) is*

- *controllable (classically) iff* $\operatorname{rank}\left(zI_n - A\,,\ B\right) = n$ *for all* $z \in \mathbb{C}$,
- *zero controllable (classically) iff* $\operatorname{rank}\left(zI_n - A\,,\ B\right) = n$ *for all* $z \in \mathbb{C} \setminus \{0\}$,

where I_n *is the* $n \times n$ *identity matrix.*

Since the right hand side of (1) may not only depend on the value of the state-like variable at time k, but also on values at times $k-1, k-2, \dots$, the classical notion of controllability may not apply. The controllability in the behavioural framework of Willems, see [14], then may be more appropriate. This notion of controllability is expressed in terms of past and future trajectories of the system that can be connected (glued) to each other by an intermediate transition trajectory to obtain a new overall trajectory that also satisfies the system equations, see also [15].

We define hereafter the notion of controllability in the behavioural sense for the type of systems that we study in the paper, see also [4] or [14].

Definition 1. System (2) is said to be controllable in behavioural sense if for all solution pairs $(x^1(k), u^1(k))$ and $(x^2(k), u^2(k))$, satisfying (2), there exists an integer $\tau \geqslant 0$ and an intermediate pair