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# Lyapunov characterization of input-to-state stability for semilinear control systems over Banach spaces☆

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ABSTRACT

We prove that input-to-state stability (ISS) of nonlinear systems over Banach spaces is equivalent to existence of a coercive Lipschitz continuous ISS Lyapunov function for this system. For linear infinite-dimensional systems, we show that ISS is equivalent to existence of a non-coercive ISS Lyapunov function and provide two simpler constructions of coercive and non-coercive ISS Lyapunov functions for input-to-state stable linear systems.



Input-to-state stability (ISS) was introduced by Sontag in his seminal paper [1] and has since become a backbone of robust nonlinear control theory. Applications of ISS include robust stabilization of nonlinear systems [2], design of nonlinear observers [3], analysis of large-scale networks [4,5] and other branches of nonlinear control [6].

The success of ISS theory of ordinary differential equations (ODEs) and the need for proper tools for robust stability analysis of partial differential equations (PDEs) motivated the development of ISS theory in the infinite-dimensional setting [7–15].

The two main lines of research within infinite-dimensional ISS theory are the development of a general ISS theory of evolution equations in Banach spaces and the application of ISS to particular important PDEs.

The results in the first area include small-gain theorems for interconnected infinite-dimensional systems and their applications to nonlinear interconnected parabolic PDEs over Sobolev spaces [8], ISS theory for linear systems over Banach spaces with admissible operators [11,16] and characterizations of local and global ISS properties [14,15]. For the second question, constructions of ISS Lyapunov functions for nonlinear parabolic systems over $L_p$-spaces [7], for linear time-variant systems of conservation laws [17], bilinear systems over Banach spaces [9], systems with saturations [18] and more have been investigated. Non-Lyapunov methods were applied to linear parabolic systems with boundary disturbances in [11,19].

In this paper, we follow the first line of research and prove converse Lyapunov theorems for ISS of linear and semilinear evolution equations in Banach spaces. For us the primary motivation comes from the papers [20,21], in which converse UGAS Lyapunov theorems have been applied to prove, in the case of ODEs, the equivalence between ISS and the existence of a smooth ISS Lyapunov function. This result along with further restatements of ISS in terms of other stability notions [21,22] and small-gain theorems [4,5] is at the heart of ISS theory.

In Section 1, we prove that *ISS is equivalent to the existence of a coercive, Lipschitz continuous ISS Lyapunov function* using the method from [21] and converse Lyapunov theorems for global asymptotic stability [23]. Along the way, we show that ISS is equivalent to the existence of a globally stabilizing feedback which is robust to multiplicative actuator disturbances of bounded magnitude.

In Section 2 we provide simpler constructions of coercive and non-coercive ISS Lyapunov functions for linear infinite-dimensional systems with bounded input operators. In particular, we show that the existence of a *non-coercive ISS Lyapunov function* is sufficient for ISS of linear systems with bounded input operators.

In Section 3 we discuss recent results and open problems in the theory of non-coercive ISS Lyapunov functions and in the Lyapunov theory for linear systems over Banach spaces with admissible input operators.

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Let $\mathbb{R}_+ := [0, \infty)$. For the formulation of stability properties the following classes of functions are useful:

$$\begin{aligned}
\mathcal{P} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous}, \\
&\qquad\qquad \gamma(r) = 0 \Leftrightarrow r = 0\}, \\
\mathcal{K} \quad &:= \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing}\}, \\
\mathcal{K}_\infty \quad &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}, \\
\mathcal{L} \quad &:= \Big\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly} \\
&\qquad\qquad \text{decreasing with } \lim_{t\to\infty} \gamma(t) = 0\Big\}, \\
\mathcal{KL} \quad &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous}, \\
&\qquad \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t \ge 0,\ \forall r > 0\}.
\end{aligned}$$

For a normed space $X$, we denote the closed ball of radius $r$ around 0 by $\overline{B}_r$ or $\overline{B}_r^X$ if we want to make the space clear.

Given normed spaces $X, W$, we call a function $f : W \to X$ locally Lipschitz continuous, if for all $r > 0$ there exists a constant $L_r$ such that

$$\|f(x) - f(y)\|_X \le L_r \|x - y\|_W \quad \forall x, y \in \overline{B}_{r,W}.$$

In the finite-dimensional case, local Lipschitz continuity is sometimes defined using neighborhoods of points, and in this case, this is of course equivalent. Note that in the infinite-dimensional case it is necessary to go to a definition on bounded balls as these are not compact. Our terminology is consistent with [24, p. 185]. This concept is called "Lipschitz continuity on bounded balls" in [25].

## 1. Converse ISS Lyapunov theorems for semilinear systems

We consider infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \tag{1}$$

where $A : D(A) \subset X \to X$ generates a strongly continuous semigroup of bounded linear operators, $X$ is a Banach space and $U$ is a normed linear space of input values. As the space of admissible inputs, we consider the space $\mathcal{U} := PC_b(\mathbb{R}_+, U)$ of globally bounded, piecewise continuous functions from $\mathbb{R}_+$ to $U$.

In this paper we consider mild solutions of (1), i.e. solutions of the integral equation

$$x(t) = T_t x(0) + \int_0^t T_{t-s} f(x(s), u(s))ds \tag{2}$$

belonging to the class $C([0, \tau], X)$ for certain $\tau > 0$. Here $\{T_t,\ t \ge 0\}$ is the $C_0$-semigroup generated by $A$. For the theory of $C_0$-semigroups and its applications to evolution equations we refer to [25,26]. In the sequel, we will write $\phi(t, x, u)$ to denote the solution of (2) corresponding to the initial condition $\phi(0, x, u) = x$ and the input $u \in \mathcal{U}$.

In the remainder of the paper we suppose that the nonlinearity $f$ satisfies the following assumption:

**Assumption 1.** Let $f : X \times U \to X$ be locally bi-Lipschitz continuous, i.e. $f$ is locally Lipschitz continuous from the normed linear space $(X \times U, \|\cdot\|_X + \|\cdot\|_U)$ to the space $X$.

Due to standard arguments, Assumption 1 implies that mild solutions corresponding to any $x(0) \in X$ and any $u \in \mathcal{U}$ exist and are unique. Bi-Lipschitz continuity of $f$ is too strong for mere existence and uniqueness, but we need it for the proof of Lemmas 1 and 4.

**Remark 1.** Note that there are interesting infinite-dimensional systems, which are not covered by the class of systems (1). In particular, boundary control systems can be described by control systems with unbounded input operators [27], which are not covered by (1). The development of converse Lyapunov results for this class of systems is a challenging problem. Some preliminary results have been shown in [28], see Section 3 for a short discussion.

Furthermore, some highly nonlinear systems (even without inputs) as e.g. the porous medium equation [29], the nonlinear KdV equation [30] are not covered by (1), and should be modeled using methods of nonlinear semigroup theory, which is closely connected to the theory of maximal monotone operators [31].

We call the system (1) forward complete, if for all initial conditions $x \in X$ and all $u \in \mathcal{U}$ the solution exists on $\mathbb{R}_+$.

We treat $u$ as an external input, which may have significant influence on the dynamics of the system. For the stability analysis of such systems a fundamental role is played by the concept of input-to-state stability, which unifies external and internal stability concepts.

**Definition 1.** System (1) is called input-to-state stable (ISS), if it is forward complete and if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that $\forall x \in X$, $\forall u \in \mathcal{U}$ and $\forall t \ge 0$ we have

$$\|\phi(t, x, u)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}). \tag{3}$$

A key tool to study ISS are ISS Lyapunov functions.

**Definition 2.** A continuous function $V : X \to \mathbb{R}_+$ is called a non-coercive ISS Lyapunov function, if $V(0) = 0$ and if there exist $\psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and $\chi \in \mathcal{K}$ so that

$$0 < V(x) \le \psi_2(\|x\|_X) \quad \forall x \in X \setminus \{0\}. \tag{4}$$

and so that the Dini derivative of $V$ along the trajectories of the system (1) satisfies the implication

$$\|x\|_X \ge \chi(\|u(0)\|_U) \quad \Rightarrow \quad \dot{V}_u(x) \le -\alpha(\|x\|_X) \tag{5}$$

for all $x \in X$ and $u \in \mathcal{U}$, where

$$\dot{V}_u(x) = \overline{\lim_{t\to+0}} \frac{1}{t}\big(V(\phi(t, x, u)) - V(x)\big). \tag{6}$$

If, in addition, there exists $\psi_1 \in \mathcal{K}_\infty$ such that

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X) \quad \forall x \in X, \tag{7}$$

then $V$ is called a coercive ISS Lyapunov function.

In Definition 2 we have defined ISS Lyapunov functions in the so-called implication form. For another (dissipative) definition of ISS Lyapunov functions and for the relation between these definitions please consult [9]. We have the following result, see [9, Proposition 1].

**Proposition 1.** *If there exists a coercive ISS Lyapunov function for (1), then (1) is ISS.*

We intend to show, see Theorem 5, that

*ISS of (1) implies the existence of a coercive, locally Lipschitz continuous Lyapunov function for (1).*

We follow the method developed in [21] for systems described by ODEs. In order to formalize the robust stability property of (1), we consider the problem of global stabilization of (1) by means of feedback laws which are subject to multiplicative disturbances with a magnitude bounded by 1. To this end let $\varphi : X \to \mathbb{R}_+$ be locally Lipschitz continuous and consider inputs

$$u(t) := d(t)\varphi(x(t)), \quad t \ge 0, \tag{8}$$

where $d \in \mathcal{D} := \{d : \mathbb{R}_+ \to D,\ \text{piecewise continuous}\}$, $D := \{d \in U : \|d\|_U \le 1\}$.

Applying this feedback law to (1) we obtain the system

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + f(x(t), d(t)\varphi(x(t))) \\
&=: Ax(t) + g(x(t), d(t)).
\end{aligned} \tag{9}$$