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State–feedback stabilization of multi-input compartmental systems

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In this paper we address the positive (state–feedback) stabilization of multi-input compartmental systems, i.e. the design of a state–feedback matrix that preserves the compartmental property of the resulting feedback system, while achieving stability. We first provide necessary and sufficient conditions for the positive stabilizability of compartmental systems whose state matrix is irreducible. Then we address the case when the state matrix is reducible, identify two sufficient conditions for the problem solution, and then extend them to a general algorithm that allows to verify when the problem is solvable and to produce a solution.

1. Introduction

The stabilization of positive systems and the dual problem of positive observer design have been the subject of several papers (see, e.g., [1–9]). Most of the literature focused on the general class of positive systems and translated the positive stabilization problem either into a Linear Matrix Inequality (LMI) [6], or into a Linear Programming (LP) problem [7], by making use of the fact that the positive/Metzler matrix of the system obtained by means of a state–feedback is Schur/Hurwitz if and only if it admits a positive diagonal Lyapunov function (condition that leads to the LMI formulation) or a linear copositive Lyapunov function (condition that leads to an LP condition). The solution in terms of LP, even if equivalent from a theoretical viewpoint, is preferable due to its lower computational complexity. Moreover, it lends itself to be easily extended to cope with robust stabilization in the presence of polytopic uncertainties, stabilization with restricted sign controls and stabilization with bounded controls [7].

Alternative approaches to the positive stabilization problem have been proposed in [8] and [2]. The characterization derived in [8] is based on the construction of certain polytopes and on verifying whether a selection of their vertices can be used to construct a stabilizing state–feedback matrix. On the other hand, in [2] the problem of achieving by means of a state–feedback not only positivity and stability, but also certain L_1 and L_∞ performance, has been investigated. Also in this case, necessary and sufficient conditions for the existence of a solution have been expressed as LPs.

(Linear) compartmental systems are a special class of positive state-space models that represent physical systems in which units, called compartments, exchange material and are subject to the law of mass conservation. Such systems were first introduced in physiology [10] and they are characterized by the fact that their state variables are nonnegative and their sum, $\sum_{i=1}^{n} x_i(t)$, cannot increase with time. For a general introduction to compartmental systems we refer the interested reader to [11,12].

The (positive) stabilization of single-input compartmental systems has been thoroughly investigated in [4] (see also [13]): very strong characterizations, that rely only on the nonzero patterns of the matrices involved in the system description, have been derived. These characterizations do not find a straightforward extension to the class of multi-input compartmental systems, for which positive stabilizability also depends on the specific entries of the involved matrices and not only on their nonzero patterns (see Example 10). On the other hand, the only results available in the multi-input case are simply the aforementioned ones, derived for the general class of multi-input positive systems. It turns out that the compartmental property allows to obtain much stronger characterizations of the positive stabilizability property. Even more, it allows to considerably simplify the LPs that provide conditions that are equivalent to the existence of a solution.

In this paper we investigate the positive stabilization of multi-input compartmental systems, by first showing that when the original system matrix A is irreducible, the positive stabilization problem is solvable if and only if it can be solved by resorting to a state–feedback that depends on a single compartment. Necessary and sufficient conditions for this to be the case are given, in the form of Linear Programming: since these conditions involve only a single column, they are quite simpler than the general ones obtained for multi-input positive systems.

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When the system matrix is reducible on the other hand, we first provide two sets of sufficient conditions for positive stabilizability that involve a very low number of system compartments, and are based on the property that a compartmental matrix is Hurwitz if and only if all its compartments are outflow connected. The intuition behind these two sufficient conditions is then formalized in graph terms and this allows to provide a necessary and sufficient condition for positive stabilization in the form of an algorithm. The algorithm provides a solution having a number of nonzero columns that does not exceed the number of communication classes of the original compartmental state matrix. This means that the solution modifies the outflow of a minimal number of compartments, specifically, at most one per communication class. From a practical point of view, this means that the state–feedback law is expressed in terms of the values of a small subset of the state variables, a property that may be extremely convenient when sensors are expensive or quite difficult to locate. For instance, it may be the case that very few state variables are actually available for measurements and in this sense the proposed solution is extremely convenient. On the other hand, in order to be able to exploit the proposed algorithm, the knowledge of the communication classes of the digraph associated with the matrix A is required. If this information is available, the algorithm can impose a significantly lower computational burden with respect to the one required to solve the LPs proposed in [7] for the general case of an unstructured pair of matrices (A, B).

The paper is organized as follows. In Section 2, notation, mathematical preliminaries and problem statement are given. In Section 3, three technical lemmas are given. As a starting point, in Section 4, the class of compartmental systems whose system matrix is irreducible is thoroughly investigated. In Section 5, the analysis is extended to address the case when the system matrix is reducible, and conditions that ensure positive stabilizability are given. Finally, Section 6 presents a necessary and sufficient condition for positive stabilization of multi-input compartmental systems in the form of an algorithm. Examples illustrate the various conditions provided in the paper. A preliminary version of the first part of this paper was presented at the IEEE Conference on Decision and Control, CDC 2017, in Melbourne, Australia [14]. In [14] we have investigated only the irreducible case and provided one sufficient condition for the solvability in the reducible case. So, the second part of Section 5 (starting from Proposition 17) and the whole Section 6, where the final problem solution and the algorithm to obtain it are derived, are the novel contribution of this paper.

2. Preliminaries and problem statement

Given $k, n \in \mathbb{Z}$, with $k \leq n$, the symbol $[k, n]_{\mathbb{Z}}$ denotes the integer set $\{k, k+1, \dots, n\}$, namely $[k, n] \cap \mathbb{Z}$. The semiring of nonnegative real numbers is denoted by $\mathbb{R}_+$. In the sequel, the (i, j)th entry of a matrix A is denoted by $[A]_{i,j}$, while the ith entry of a vector $\mathbf{v}$ is denoted by $[\mathbf{v}]_i$. Following [15], we adopt the following terminology and notation. Given a matrix A with entries $[A]_{i,j}$ in $\mathbb{R}_+$, we say that A is a *nonnegative matrix*, if all its entries are nonnegative, namely $[A]_{i,j} \geq 0$ for every i, j, and if so we use the notation $A \geq 0$. If A is a nonnegative matrix, and $A \neq 0$, then A is said to be a *positive matrix* and we adopt the notation $A > 0$. Note that $A > 0$ does not mean that all its entries are positive, but simply that at least one of them is positive and the remaining ones are nonnegative. Notation $A \geq B$ $(A > B)$ means $A - B \geq 0$ $(A - B > 0)$. The symbols $\leq$ and $<$ are defined accordingly. Also, the same notation is adopted for vectors.

We let $\mathbf{e}_i$ denote the ith vector of the canonical basis in $\mathbb{R}^n$ (where n is always clear from the context), whose entries are all zero except for the ith one that is unitary. The symbol $\mathbf{1}$ denotes a vector with all entries equal to 1 (and whose size is clear from the context). Given a matrix $A \in \mathbb{R}^{n\times m}$ (in particular, a vector), its *nonzero pattern* $\overline{\mathrm{ZP}}(A)$ is the set $\{(i, j) \in [1, n]_{\mathbb{Z}} \times [1, m]_{\mathbb{Z}} : [A]_{i,j} \neq 0\}$. We denote by $S_i \in \mathbb{R}^{(n-1)\times n}$ the selection matrix obtained by removing the ith row in the identity matrix I_n, namely

$$S_i = \left[\begin{array}{c|c|c} I_{i-1} & 0 & 0_{(i-1)\times(n-i)} \\ \hline 0_{(n-i)\times(i-1)} & 0 & I_{n-i} \end{array}\right].$$

The size n will always be clear from the context, namely from the size of the matrix or vector S_i is applied to. For any matrix $A \in \mathbb{R}^{n\times m}$, S_iA denotes the matrix obtained from A by removing the ith row, while for any vector $\mathbf{v} \in \mathbb{R}^n$, $S_i\mathbf{v}$ is the vector obtained from $\mathbf{v}$ by removing the ith entry. A real square matrix A is *Hurwitz* if all its eigenvalues lie in the open left complex halfplane, i.e. for every λ belonging to the spectrum $\sigma(A)$ of A we have $\mathrm{Re}(\lambda) < 0$.

A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. This is equivalent to saying that for every $h \in [1, n]_{\mathbb{Z}}$ the vector $S_hA\mathbf{e}_h$ is nonnegative. For $n \geq 2$, an $n\times n$ nonzero Metzler matrix A is *reducible* [16,17] if there exists a permutation matrix Π such that

$$\Pi^\top A\Pi = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ and $A_{2,2}$ are square (nonvacuous) matrices, otherwise it is *irreducible*. In general, given a Metzler matrix A, a permutation matrix Π can be found such that

$$\Pi^\top A\Pi = \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,s} \\ 0 & A_{2,2} & \dots & A_{2,s} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{s,s} \end{bmatrix}, \quad (1)$$

where each diagonal block $A_{i,i}$, of size $n_i\times n_i$, is either scalar ($n_i = 1$) or irreducible. Eq. (1) is usually referred to as *Frobenius normal form* of A [17,18].

If A is an $n \times n$ Metzler matrix, then as proved in [19] it exhibits a real dominant (not necessarily simple) eigenvalue, known as *Frobenius eigenvalue* and denoted by $\lambda_F(A)$. This means that $\lambda_F(A) > \mathrm{Re}(\lambda)$, $\forall\, \lambda \in \sigma(A)$, $\lambda \neq \lambda_F(A)$.

Basic definitions and results about cones may be found, for instance, in [20,21]. We recall here only those facts that will be used within this paper. A set $\mathcal{K} \subset \mathbb{R}^n$ is said to be a *cone* if $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$; a cone is *convex* if it contains, with any two points, the line segment between them. A convex cone $\mathcal{K}$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of *generating vectors*. This means that a positive integer k and a matrix $W \in \mathbb{R}^{n\times k}$ can be found, such that $\mathcal{K}$ coincides with the set of nonnegative combinations of the columns of W. In this case, we adopt the notation $\mathcal{K} := \mathrm{Cone}(W)$. A convex cone $\mathcal{K}$ is polyhedral if and only if there exists a matrix $C \in \mathbb{R}^{p\times n}$ such that $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n : C\mathbf{x} \geq 0\}$.

A Metzler matrix endowed with the additional property that the entries of each of its columns sum up to a nonpositive number, i.e., $\mathbf{1}^\top A \leq 0^\top$, is called *compartmental matrix* (see [11,22]). For any such matrix the Frobenius eigenvalue $\lambda_F(A)$ is nonpositive, and if $\lambda_F(A) = 0$ then A is simply stable, i.e., it has the constant mode associated with $\lambda_F(A) = 0$, but no unstable modes.

In this paper we will focus on compartmental models, which are typically used to describe material or energy flows among compartments of a system. Each compartment represents a homogeneous entity within which the entities being modelled are equivalent. An n-dimensional *multi-input linear* compartmental system is a linear state-space model

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad (2)$$