Systems & Control Letters 118 (2018) 1–7

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Stability and performance of discrete-time switched linear systems[☆]

Grace S. Deaecto[a,*], José C. Geromel[b]

[a] DMC/School of Mechanical Engineering, UNICAMP, 13083-860, Campinas, SP, Brazil
[b] DCA/School of Electrical and Computer Engineering, UNICAMP, 13083 - 852, Campinas, SP, Brazil

ARTICLE INFO

Article history:
Received 24 April 2017
Received in revised form 29 January 2018
Accepted 15 May 2018

Keywords:
Switched linear systems
Stability analysis
$\mathcal{H}_2$ control design

ABSTRACT

This paper deals with stability analysis and control design of discrete-time switched linear systems. The results are based on a new sufficient condition for exponential stabilizability. A performance index that falls in the context of $\mathcal{H}_2$ norm is considered in order to optimize the joint design of a state dependent switching function and a state feedback control law. All control design conditions are expressed through linear matrix inequalities (LMIs). Comparisons with other available design procedures are made by means of examples borrowed from the literature. The present procedure is more amenable for control synthesis purposes and simpler from both theoretical and numerical viewpoints. Three state feedback control switching strategies are presented. The complexity of the control law is discussed.

1. Introduction

Switched systems have been widely studied in the last decades due to their intrinsic importance from both theoretical and practical viewpoints. The books [1], and [2] contain complete and useful results in the general framework of switched systems analysis and control design. In addition, the survey papers [3,4] and [5] and the references therein have put in evidence problems and new theoretical aspects that due to the lack of a complete solution needed to be further considered.

In this general context, a switched system can be viewed as a dynamical system constituted of a number of subsystems and a rule that orchestrates the switching among them, in other words, a logical strategy that decides the activation of a specific subsystem at each instant of time. There are two main classes characterized by the nature of the switching rule. Indeed, it can be a perturbation or a control variable to be designed with specific goals of stability and performance enhancement.

Several contributions on both classes are available in the literature. For the first class Refs. [6,7] provide sufficient conditions for stability based on multiple Lyapunov functions assuming arbitrary and bounded dwell time perturbations. This path was completely characterized by the necessary and sufficient conditions for stability and performance optimization provided in [8] and [9]. The main characteristic of these results is that they are well adapted for control systems purpose since they are expressed in terms of LMIs [10]. For the second class, almost the same has occurred. Sufficient conditions for the existence of a stabilizing switching rule (state or output dependent) have been proposed in Refs. [11] for switched affine systems, [12] for sampled-data switched linear systems, [13,14], and [15] for switched linear systems in discrete-time, among many others. It is worth mentioning the importance of the sufficient conditions proposed in [15] where the role of the convex combination has been pointed out for the first time in the context of discrete-time switched linear systems. Naturally, in this vein, necessary and sufficient conditions for stabilizability [16] and [17] and LQR performance optimization [18–20], and [21] also appeared.

This paper proposes new sufficient conditions for exponential stabilizability of discrete-time switched linear systems based on a quadratic but time-varying Lyapunov function. The conditions are expressed by linear matrix inequalities and, therefore, are simpler to solve than others available in the literature. The advantage of the present stabilizability conditions when compared to the cited ones is that, due to convexity, they apply to switching function and state feedback control joint design and can be generalized to cope with output feedback control and filtering. Moreover, on the contrary of [13], our approach is able to deal with performance optimization, being well adapted to handle $\mathcal{H}_2$ and $\mathcal{H}_\infty$ indexes, see [22] for details.

The notation used throughout is standard. For square matrices, $\mathbf{tr}(\cdot)$ denotes the trace function. For real vectors or matrices, $(')$ refers to their transpose. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. For any real symmetric matrix, $X > 0$ $(X \geq 0)$ denotes a positive (semi)definite matrix. The set composed of the N first positive natural numbers is denoted by

☆ This research was supported by the Brazilian National Research Council - CNPq (Grant numbers 303887/2014-1 and 306911/2015-9) and by São Paulo Research Foundation - FAPESP (Grant number 2017/20343-0).
* Corresponding author.
E-mail addresses: grace@fem.unicamp.br (G.S. Deaecto), geromel@dsce.fee.unicamp.br (J.C. Geromel).

https://doi.org/10.1016/j.sysconle.2018.05.006
0167-6911/

$\mathbb{K} = \{1, \ldots, N\}$. The unit simplex composed by all nonnegative vectors $\lambda \in \mathbb{R}^N$ such that $\sum_{j\in\mathbb{K}} \lambda_j = 1$ is denoted by Λ. The ith column of any identity matrix is denoted by e_i. The square norm of a trajectory $z(n),\ n \in \mathbb{N}$ is $\|z\|_2^2 = \sum_{n\in\mathbb{N}} \|z(n)\|^2$ where $\|z(n)\|^2 = z(n)'z(n)$ is the square of the Euclidean norm. The greatest integer less or equal to a is $\lfloor a \rfloor$. A square matrix is said Schur stable if its eigenvalues belong to the open region $|z| < 1$ of the complex plane.

2. Problem statement

Consider a discrete-time switched linear system

$$x(n+1) = A_\sigma x(n) + E_\sigma w(n) \tag{1}$$

$$z(n) = C_\sigma x(n) \tag{2}$$

where $x(\cdot) \in \mathbb{R}^{n_x}$, $w(\cdot) \in \mathbb{R}^{n_w}$ and $z(\cdot) \in \mathbb{R}^{n_z}$, defined for all $n \in \mathbb{N}_- = \mathbb{N} \cup \{-1\}$, are the state, the exogenous input, and the controlled output, respectively. It is assumed that (1)–(2) evolves, for all $n \in \mathbb{N}_-$, from null[1] initial condition $x(-1) = 0$. The control action is accomplished by means of a switching function $\sigma(n) : \mathbb{N}_- \to \mathbb{K}$ which, in some instances, may be state dependent, that is, $\sigma(n) = v(x(n))$ for some function $v(\cdot) : \mathbb{R}^{n_x} \to \mathbb{K}$. For a given performance index, the goal is to determine a suboptimal switching function to the control design problem

$$\inf_{\sigma\in\mathcal{V}} J(\sigma) \tag{3}$$

where $\mathcal{V}$ is the set of all switching policies assuring that the closed-loop system (1)–(2) is globally stable. The following definition, presented in [13], will be useful in the sequel.

Definition 1. The switched linear system $x(n+1) = A_\sigma x(n)$ is exponentially stabilizable if there exist constants $c \geq 0, 0 \leq \mu < 1$ and for each $x(0) \in \mathbb{R}^{n_x}$ a switching trajectory $\{\sigma(n)\}_{n\in\mathbb{N}}$ such that $\|x(n)\| \leq c\mu^n \|x(0)\|$ for all $n \in \mathbb{N}$.

It is important to keep in mind that in the context of exponential stability different switching functions $\{\sigma(n)\}_{n\in\mathbb{N}}$ can be adopted for different initial conditions $x(0) \in \mathbb{R}^{n_x}$. The next definition is more stringent by imposing only one switching trajectory for all initial conditions.

Definition 2. The switched linear system $x(n+1) = A_\sigma x(n)$ is Schur stabilizable if there exist constants $c \geq 0, 0 \leq \mu < 1$ and a switching trajectory $\{\sigma(n)\}_{n\in\mathbb{N}}$ such that $\|x(n)\| \leq c\mu^n \|x(0)\|$ for all $x(0) \in \mathbb{R}^{n_x}$ and all $n \in \mathbb{N}$.

If Schur stabilizability holds for some $\{\sigma(n)\}_{n\in\mathbb{N}}$ then it also holds for the periodic switching law $\{\sigma_p(n)\}_{n\in\mathbb{N}}$ with period $\kappa \in \mathbb{N}$ large enough, such that $c\mu^\kappa < 1$ and $\sigma_p(n) = \sigma(n)$ for $n \in [0, \kappa)$. Hence, in this case, periodic stabilizability as defined in [13] also holds.

The validity of Definition 2 can be tested by $x(n+1) = A_\sigma x(n)$ with a periodic switching law. Consequently, Schur stabilizability holds if and only if there exists $0 < \kappa \in \mathbb{N}$ finite such that for some $\{\sigma(n)\}_{n=0,\ldots,\kappa-1}$ the matrix product $\mathbb{A}_{(0,\kappa-1)} = A_{\sigma(\kappa-1)} \cdots A_{\sigma(1)} A_{\sigma(0)}$ is Schur stable. The fact that $\kappa \geq 1$ may be taken arbitrarily large (but finite) puts in evidence that although not equivalent both stability concepts provided by Definitions 1 and 2 are closely related. We are now in position to introduce a performance index that is similar to the celebrated $\mathcal{H}_2$ norm of linear time-invariant systems. It is well-defined whenever the switching function makes the closed-loop switched system exponentially stable, that is $\sigma \in \mathcal{V}$.

Definition 3. The $\mathcal{H}_2$ performance index associated with the closed-loop switched linear system (1)–(2) with initial condition $x(-1) = 0$ is given by

$$J(\sigma) = \sum_{r=1}^{n_w} \|z_r\|_2^2 \tag{4}$$

where $z_r(n),\ n \in \mathbb{N}_-$, is the output of the system corresponding to the impulsive input $w(n) = e_r\delta(n+1)$, for all $r = 1, \ldots, n_w$.

Simple algebraic manipulations put in evidence that this performance index can alternatively be calculated from (1)–(2) with null exogenous input $w(n) = 0,\ \forall n \in \mathbb{N}$, and initial conditions $x(0) = E_{\sigma(-1)} e_r$ for all $r = 1, \ldots, n_w$. As it was already commented, it is important to keep in mind that in Definition 3, the initial condition as well as the impulse perturbation have been displaced by one time interval in order to maintain this property true so as the performance index becomes simpler to be determined. It is immediate to see that the LQR index considered in [20] and [21] can be recast in the context of Definition 3.

2.1. Illustrative example 1

This is an example borrowed from Ref. [7] used to illustrate some relevant aspects of the system under consideration. It consists of two second order subsystems defined by matrices $A_i = e^{A_{ci}T},\ i = 1, 2$, with $T = 0.1$ and

$$A_{c1} = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix},\ A_{c2} = \begin{bmatrix} 0 & 1 \\ -2 & 8 \end{bmatrix}.$$

Matrices A_1 and A_2 are not Schur stable but the matrix product $\mathbb{A}_{(0,1)} = A_2A_1$ of length $\kappa = 2$ is. The eigenvalues of $\mathbb{A}'_{(0,1)}\mathbb{A}_{(0,1)}$ are $\{0.7586, 1.0792\}$. This means that the constraint $\mathbb{A}'_{(0,1)} S \mathbb{A}_{(0,1)} < S$ is satisfied for some matrix $S > 0$ but it does not admit $S = I$ as a solution. The first integer $\ell \in \mathbb{N}$ such that $\mathbb{A}^{\ell'}_{(0,1)}\mathbb{A}^{\ell}_{(0,1)} < I$ is $\ell = 148$. Hence, there exists $\mathbb{A}_{(0,295)}$ such that $\mathbb{A}'_{(0,295)}\mathbb{A}_{(0,295)} < I$ which is a stability test similar to the one proposed in [13]. In this example, the effect of matrix S is very important to reduce the length of the matrix product.

3. Switching control design

In this section, a switched linear system with null exogenous input $w(n) = 0,\ \forall n \in \mathbb{N}$, and arbitrary initial condition $x(0) = x_0 \in \mathbb{R}^{n_x}$ is considered. In this case, the switched linear system (1)–(2) reduces to

$$x(n+1) = A_\sigma x(n),\ x(0) = x_0 \tag{5}$$

$$z(n) = C_\sigma x(n) \tag{6}$$

for all $n \in \mathbb{N}$. For a given $\kappa \geq 1$, the sequence $\{m_n\}_{n\in\mathbb{N}}$ with generic term $m_n = \kappa\lfloor n/\kappa \rfloor$ indicates the index of the first element of the $(\lfloor n/\kappa \rfloor + 1)$th subsequence of length κ in the interval $[0, n]$ for all $n \in \mathbb{N}$. It is used to establish a sufficient condition that yields a switching function assuring exponential stability and norm bounded performance.

3.1. Stabilizing control design

The stability analysis and control design conditions to be presented afterwards are based on the next theorem and corollary that constitute the main results of this paper.

[1] It is simple to verify that the effect of an impulsive input at $n = -1$ together with some given initial condition $x(-1)$ and $\sigma(-1)$ can be converted to an initial condition $x(0)$. This adjustment is adopted with no loss of generality only to ease presentation. This is necessary because the system (1)–(2) is time-varying, see [14].