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Small-gain stability analysis of certain hyperbolic–parabolic PDE loops

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ABSTRACT

This work provides stability results in the spatial sup norm for hyperbolic–parabolic loops in one spatial dimension. The results are obtained by an application of the small-gain stability analysis. Two particular cases are selected for the study because they contain challenges typical of more general systems (to which the results are easily generalizable but at the expense of less pedagogical clarity and more notational clutter): (i) the feedback interconnection of a parabolic PDE with a first-order zero-speed hyperbolic PDE with boundary disturbances, and (ii) the feedback interconnection, by means of a combination of boundary and in-domain terms, of a parabolic PDE with a first-order hyperbolic PDE. The first case arises in the study of the movement of chemicals underground and includes the wave equation with Kelvin–Voigt damping as a subcase. The second case arises when we apply backstepping to a pair of hyperbolic PDEs that is obtained by ignoring diffusion phenomena. Moreover, the second case arises in the study of parabolic PDEs with distributed delays. In the first case, we provide sufficient conditions for ISS in the spatial sup norm with respect to boundary disturbances. In the second case, we provide (delay-independent) sufficient conditions for exponential stability in the spatial sup norm.

1. Introduction

The use of the notion of Input-to-State Stability (ISS) for finite-dimensional systems, which was proposed by E.D. Sontag in [1], allowed the development of small-gain theorems. Starting with the first nonlinear, generalized small-gain theorem in [2] for systems described by Ordinary Differential Equations (ODEs), the small-gain stability analysis has been extended to various kinds of systems with inputs (see [3]). The extension of ISS to systems described by Partial Differential Equations (PDEs; see [4–6]) allowed the application of small-gain arguments in [7] for systems of interconnected PDEs. The recent extension of ISS to PDEs with boundary disturbances in [8,9] allowed the use of small-gain arguments in [9] to PDEs with non-local boundary conditions. The use of small-gain arguments in [9] also showed that small-gain analysis is capable of providing stability estimates in the spatial sup norm. This feature can rarely be met in Lyapunov analysis (which is more well-suited for estimates in L^p spatial norms with $2 \leq p < +\infty$).

The study of interconnected PDEs arises naturally in many applications. Interconnections of PDEs have been studied in [10]. The literature focuses on the study of systems of hyperbolic PDEs (see [11–14]) and Reaction–Diffusion systems (i.e., systems of parabolic PDEs; see for instance [15]). ODE–PDE cascades have been studied in [16–20], mostly for feedback and observer design purposes. However, the present work is devoted to the study of parabolic–hyperbolic PDE loops. Such loops present unique features because they combine the finite signal transmission speed of hyperbolic PDEs with the unlimited signal transmission speed of parabolic PDEs. Since there are many possible interconnections that can be considered, it is difficult to give results for a "general case". Therefore, we focus on two particular cases, which are analyzed in detail, because they contain challenges typical of more general systems (to which the results are easily generalizable but at the expense of less pedagogical clarity and more notational clutter).

The first case considered in this paper is the feedback interconnection of a parabolic PDE with a special first-order hyperbolic PDE: a zero-speed hyperbolic PDE. Thus the action of the hyperbolic PDE resembles the action of an infinite-dimensional, spatially parameterized ODE. However, the study of this particular loop is of special interest because it arises in an important application: the movement of chemicals underground ([21], pages 210–216). Moreover, the study of this particular system can be used for the analysis of wave equations with Kelvin–Voigt damping (see also [22–25]). In this case, we provide sufficient conditions for ISS in the spatial sup norm with respect to boundary disturbances (Theorem 2.2 and Corollary 2.3). There are no available stability results in the literature for the wave equation with Kelvin–Voigt damping in the spatial sup norm (even when boundary disturbances are absent).

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The second case considered in this paper is the feedback interconnection, by means of a combination of boundary and in-domain terms, of a parabolic PDE with a first-order hyperbolic PDE. The interconnection is effected by linear, non-local terms. The second case arises when we apply backstepping to a pair of hyperbolic PDEs that is obtained by ignoring diffusion phenomena (see [26,27]). Moreover, the second case arises in the study of parabolic PDEs with distributed delays of trace terms. Parabolic equations with delayed terms have also been studied in [28,29]. In this case, we provide sufficient conditions for exponential stability in the spatial sup norm with respect to boundary disturbances (Theorem 2.6). This is an important result for control purposes, because it shows that boundary controllers designed with the backstepping methodology are robust with respect to diffusion (which is a high-order perturbation term). The obtained result is independent of the speed of the hyperbolic PDE and can be interpreted as a delay-independent stability condition for the corresponding parabolic PDE with delayed trace terms.

The present work is structured as follows: the main results of the paper are stated in Section 2. Section 3 contains the proofs of all results. Finally, the concluding remarks of the present work are presented in Section 4.

2. Main results

2.1. Movement of chemicals underground

Certain chemicals are released at position $\xi = 0$ and enter the groundwater system. Let $\phi \in (0,1)$ be the porosity of the soil, $v \geq 0$ be the velocity of the bulk movement of the groundwater, $c(t,\xi)$ and $n(t,\xi)$ be the concentration of the chemicals dissolved in water and the sorbed concentration of chemicals in the soil, respectively, at position $\xi \in [0,L]$ (horizontal coordinate) and time $t \geq 0$.

The physical law that allows us to obtain a mathematical model for this process is Fick's law: the rate per unit area per unit time that mass of chemicals crosses a plane section through the flow at position $\xi \in [0,L]$ and time $t \geq 0$ is equal to $-D\frac{\partial c}{\partial \xi}(t,\xi)+\phi\, v\, c(t,\xi)$, where $D > 0$ is the diffusion coefficient. Taking into account that the rate of sorption of chemicals in the soil is proportional to $\frac{\partial n}{\partial t}(t,\xi)$, the mass balance for the chemical gives the following equation:

$$\frac{\partial c}{\partial t}(t,\xi) = \frac{D}{\phi}\frac{\partial^2 c}{\partial \xi^2}(t,\xi) - v\frac{\partial c}{\partial \xi}(t,\xi) - \frac{1}{\phi}\frac{\partial n}{\partial t}(t,\xi), \text{ for } (t,\xi) \in (0,+\infty)\times(0,L). \tag{2.1}$$

We assume that the concentration of chemicals dissolved in underground water at $\xi = 0$ is time-varying and takes values around a nominal value $c_0 > 0$. At $\xi = L$ the ground meets the sea, where the concentration of chemicals is zero. Therefore, we obtain the boundary conditions:

$$\begin{aligned} c(t,0) &= c_0 + \tilde{d}(t) \\ c(t,L) &= 0 \end{aligned}, \text{ for all } t \geq 0 \tag{2.2}$$

where $\tilde{d}$ is the variation of the concentration of chemicals dissolved in water at the source ($\xi = 0$).

In order to complete the description of the mathematical model of the process, we need an empirical relation that provides quantitative information about the rate of sorption of chemicals in the soil. The rate of sorption of chemicals in the soil at position $\xi \in [0,L]$ and time $t \geq 0$ has to be a non-decreasing function of $c(t,\xi)$ and a non-increasing function of $n(t,\xi)$. The simplest relation that describes such a dependence is

$$\frac{\partial n}{\partial t}(t,\xi) = a\,c(t,\xi) - bn(t,\xi), \text{ for } (t,\xi) \in (0,+\infty)\times(0,L) \tag{2.3}$$

where $a, b > 0$ are constants.

Substituting (2.3) into (2.1) and defining

$$u_1(t,z) := c_0^{-1}\exp\left(-\frac{vL\phi}{2D}z\right)\left(c\left(\frac{L^2\phi}{D}t, Lz\right) - c_{eq}(Lz)\right),$$

$$u_2(t,z) := c_0^{-1}\exp\left(-\frac{vL\phi}{2D}z\right)\left(n\left(\frac{L^2\phi}{D}t, Lz\right) - n_{eq}(Lz)\right),$$

to be the scaled deviations from the nominal concentration profiles $c_{eq}(\xi) = \frac{b}{a}n_{eq}(\xi) = c_0\frac{\exp\left(\frac{\phi v}{D}L\right)-\exp\left(\frac{\phi v}{D}\xi\right)}{\exp\left(\frac{\phi v}{D}L\right)-1}$ for $\xi \in [0,L]$, we obtain from (2.1), (2.2) and (2.3) the following mathematical model of the process:

$$\begin{aligned} \frac{\partial u_1}{\partial t}(t,z) &= \frac{\partial^2 u_1}{\partial z^2}(t,z) - K\,u_1(t,z) + r\tilde{b}u_2(t,z) \\ \frac{\partial u_2}{\partial t}(t,z) &= \tilde{a}u_1(t,z) - \tilde{b}u_2(t,z) \end{aligned}$$
$$\text{for } (t,z) \in (0,+\infty)\times(0,1) \tag{2.4}$$

$$\begin{aligned} u_1(t,0) &= d(t) \\ u_1(t,1) &= 0 \end{aligned}, \text{ for } t \geq 0 \tag{2.5}$$

where $d(t) := c_0^{-1}\tilde{d}\left(\frac{L^2\phi}{D}t\right)$, $\tilde{a} := a\frac{L^2\phi}{D}$, $\tilde{b} := b\frac{L^2\phi}{D}$, $r := \phi^{-1}$, $K := L^2\frac{v^2\phi^2+4aD}{4D^2}$. All parameters and variables appearing in model (2.4), (2.5) are dimensionless.

System (2.4), (2.5) is the feedback interconnection of a parabolic PDE with a first-order zero-speed hyperbolic PDE (or an infinitely-parameterized scalar ODE). Its dynamical behavior is very different from that of a parabolic PDE: to see this notice that system (2.4), (2.5) may be transformed to a wave equation (or Klein–Gordon equation) with Kelvin–Voigt damping that may also include viscous damping and stiffness terms

$$\begin{aligned} &\frac{\partial^2 u_1}{\partial t^2}(t,z) + (\tilde{b}+K)\frac{\partial u_1}{\partial t}(t,z) \\ &= \frac{\partial^3 u_1}{\partial z^2 \partial t}(t,z) + \tilde{b}\frac{\partial^2 u_1}{\partial z^2}(t,z) + \tilde{b}\,(r\tilde{a} - K)\,u_1(t,z), \\ &\text{for } (t,z) \in (0,+\infty)\times(0,1) \end{aligned}$$

and, conversely, any wave (or Klein-Gordon) equation with Kelvin–Voigt damping can be transformed to the form (2.4), (2.5).

In what follows, for a given $u : \Re_+ \times [0,1] \to \Re$ we use the notation $u[t]$ to denote the profile at certain $t \geq 0$, i.e., $(u[t])(x) = u(t,x)$ for all $x \in [0,1]$. We next provide existence/uniqueness results for the initial–boundary value (2.4), (2.5) with

$$u_1[0] = u_{1,0},\ u_2[0] = u_{2,0} \tag{2.6}$$

where $u_{1,0}, u_{2,0}$ are real functions on $[0,1]$. Our main result is the following theorem.

Theorem 2.1 (*Existence/Uniqueness*)**.** *Consider the initial-boundary value problem (2.4), (2.5), (2.6), where $K, r, \tilde{a}, \tilde{b} \in \Re$ are constants. For every $u_{2,0} \in C^1([0,1])$, $u_{1,0} \in \left\{w \in H^3(0,1) : w(1) = w''(0) = w''(1) = 0\right\}$ and for every disturbance input $d \in C^2(\Re_+)$ with $d(0) = u_{1,0}(0)$, there exists a unique pair of mappings $u_1 \in C^0(\Re_+ \times [0,1]) \cap C^1((0,+\infty)\times[0,1])$, $u_2 \in C^1(\Re_+ \times [0,1])$ with $u_1[t] \in C^2([0,1])$ for $t > 0$ satisfying (2.4), (2.5), (2.6).*

Theorem 2.1 implies that there exists a set of initial conditions $S \subseteq \left\{w \in C^2([0,1]) : w(1) = 0\right\} \times C^1([0,1])$ with the following property:

> "For every $u_0 = (u_{1,0}, u_{2,0}) \in S$ there exists a non-empty set $\Phi(u_0) \subseteq \left\{d \in C^0(\Re_+) \cap C^1((0,+\infty)) : d(0) = u_{1,0}(0)\right\}$ such that for every disturbance input $d \in \Phi(u_0)$ the initial-boundary value problem (2.4), (2.5), (2.6), there exists a unique pair of